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www.uf.bg.ac.rs; e-mail: inovacije@uf.bg.ac.rs
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University of Belgrade, Teacher Education Faculty, Kraljice Natalije 43, Belgrade, Serbia
www.uf.bg.ac.rs; e-mail: inovacije@uf.bg.ac.rs
Telephone: + 381 11 3615225, ext. 128; Fax: + 381 11 2641060

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Teaching Innovations

Word of editor-in-chief

Dear colleagues,

It is our great pleasure and honour to invite you to be our associates – authors and reviewers of scientific and research papers in the *Teaching Innovations* periodical, issued by the University of Belgrade, Teacher Education Faculty. The fact that our periodical has been published for thirty years, its current rating (categorised as M52 in the list of scientific publications of the Ministry of Education, Science and Technological Development of the Republic of Serbia) and the intention of the new editorial board to further improve its rating through the quality of papers show that the periodical *Teaching Innovations* has a long tradition based on the qualities of continuity and actuality, and a potential to continue developing.

The Teaching Innovations periodical will be publishing systematic and original research papers related to sciences and scientific disciplines dealing with the teaching process at all levels of pedagogical and educational work (from pre-school pedagogical work to life-long learning) with the aim of its improvement and modernisation.

General information about the Periodical with the Instructions for the authors and standards for paper preparation are placed on official website of Teacher Education Faculty, University of Belgrade (http://www.uf.bg.ac.rs/?page_id=13195).

Please note that the Periodical will be available in the electronic form (at the site of the Teacher Education Faculty in Belgrade) starting from issue No. 1/2014.

Looking forward to successful cooperation,
Sincerely Yours,
Vera Ž. Radović, PhD,
Editor-in-chief

Реч уредника

Поштоване колегинице, поштоване колеге,

Част нам је и задовољство да Вас позовемо да будете наши сарадници – аутори и рецензенти научних и стручних радова у часопису *Иновације у настави*, који издаје Учитељски факултет Универзитета у Београду. Чињеница да је од оснивања часописа протекло тридесет година, његов садашњи рејтинг (на листи је научних публикација Министарства просвете, науке и технолошког развоја РС у категорији М52) и настојање новог уређивачког одбора да квалитетом радова тај рејтинг подигне указују на то да часопис *Иновације у настави* има дугу традицију, да су континуитет и актуелност његови квалитети, а свакако показује како он поседује потенцијал да и у будућности напредује.

У *Иновацијама* ћемо објављивати прегледне и оригиналне истраживачке радове из наука и научних дисциплина које третирају наставни процес на свим нивоима васпитања и образовања (од предшколског васпитања до целоживотног образовања) у циљу његовог унапређења и модернизације.

Опште информације о часопису са Упутством за ауторе и стандардима за припрему рада налазе се на сајту Учитељског факултета у Београду (http://www.uf.bg.ac.rs/?page_id=13195).

Обавештавамо Вас да ће од броја 1/2014 часопис бити доступан и у електронској форми (на сајту Учитељског факултета у Београду).

Са вером у успешну сарадњу,
Срдачан поздрав,
др Вера Ж. Радовић
главни и одговорни уредник

HISTORY OF MATHEMATICS IN EDUCATION AND HISTORY OF MATHEMATICS EDUCATION – MATHEMATICAL EDUCATION CULTURES

Issue Editors:

Snezana Lawrence, PhD, Guest Editor
Bath Spa University, School of Education, Bath, UK

Olivera Djokić, PhD
University of Belgrade, Teacher Education Faculty, Belgrade, Serbia

ИСТОРИЈА МАТЕМАТИКЕ У ОБРАЗОВАЊУ И ИСТОРИЈА МАТЕМАТИЧКОГ ОБРАЗОВАЊА – КУЛТУРА МАТЕМАТИЧКОГ ОБРАЗОВАЊА

Уредници Темата:

др Снежана Лоренс, гост уредник
Педагошки факултет, Универзитет Бат Спа, Велика Британија

др Оливера Ђокић
Учитељски факултет, Универзитет у Београду, Србија

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Word of guest editor

The history of mathematics and its uses of in mathematics education have been identified and described many times in the last century or so. More recently, in the past decade, they have been classified and identified both in terms of the uses in the classroom, and the uses of the history of mathematics in mathematics teacher education and training. The various international bodies now give attention to the history of mathematics in and of education:

1. ICME (International Congress of Mathematics Education) has a regular topic study groups – one on the history of mathematics in education and one on the history of mathematics of education
2. HPM (History and Pedagogy of Mathematics) is a world-wide association of academics who hold biannual meetings; once every four years meetings are held as satellite meetings to ICME and in between those (two years into the period) it holds European Summer University. The HPM is an associate of the International Mathematics Union and contributes to the work of the same.
3. The national academic associations such as the Canadian Society for the History of Mathematics and the British Society for the History of Mathematics hold regular (and sometimes joint) meetings which link colleagues from across the Atlantic.

All of these events generate a considerable scholarship which finds publishing outlets in proceedings and in addition the three journals in the English speaking world that regularly deal with the matters of interest to us here: the *Historia Mathematica*, the *BSHM Bulletin (Journal of the British Society for the History of Mathematics)*, and the *International Journal for the History of Mathematics Education*. So why do yet another special edition or volume such as the one we have in front of us here?

Two main reasons come to mind. Redefining identities that have emerged since the collapse of the Berlin Wall relate also to the redefining influences and ways of communicating educational policies. The history of mathematics, and the history of mathematics education, although primarily concerned with mathematics itself, are nevertheless coloured by the national and international contexts of societies from which they arise, and this issue testifies to that through different interests and foci of some of the papers.

The second reason is the learning itself: learning of the children in our classrooms, teachers in training and development, and our own learning. Our is the crucial world here – our contributors' and the contributors' societies learning, is very much the focus of our efforts – through this issue we look at the practices and traditions of societies that span the international mathematics education community from Japan and Indonesia, via historical empires, to the present-day Catalonia and Serbia. But the selection itself also shows the interest that these contributors were keen to engage with the process of communicating their experiences and knowledge via the first English edition of the *Innovations*, dedicated to the teaching of mathematics.

This is then the place where we mention each of the contributions and thank the contributors for their efforts, professionalism, extremely interesting stories, and their support for this new initiative.

The first in line is 'History of mathematics and teaching mathematics' by **Mirko Dejić** and **Aleksandra M. Mihalović**, whose paper gives interesting quantitative and qualitative data which is an outcome of a wide-survey of teachers and their uses of the history of mathematics in their classroom. It is a great introduction to the research in mathematics education that shows the links between competencies of mathematics teachers and their awareness of the use of history of mathematics in primary classrooms.

On the 'Culture of Creativity in Mathematics Education' by **Bronislaw Czarnocha** gives a fascinating account of the ways in which teachers can support creativity in the pupils, and these are not only described but actually given in a set of principles to be used in the teaching of mathematics. Czarnocha shows us some of the work he has developed and presented at the 38th meeting of the International Group for the *Physchology of Mathematics Education* (PME) earlier in 2014 (July 15-20).

The paper that gives an overview of mathematics education in the **Balkan societies** I wrote based on the research I did some years ago for my chapter on the Balkan mathematics which appeared in the Oxford *Handbook for History of Mathematics*. This gives an insight into the historical framework from which Balkan societies developed their educational practices. Both Greek and Ottoman mathematics is, in the view of the contemporary mathematician perhaps seen as the origin of two greatest mathematical traditions, but their mathematical education was heavily influenced and dependent on developments from the Western European countries in the 19th century. The Serbian mathematics education is quite unique – to create a strong mathematical culture from one school and virtually one mathematician shows some ingenuity and resilience of spirit.

Mailizar Mailizar, **Manahel Alafeleq**, and **Linguo Fan**'s article gives us an overview of the historical overview of mathematics curriculum reform and development in modern Indonesia. One of the largest educational systems of the world, but little known outside of the country, and certainly in Europe, this is an interesting insight into the South-East Asian mathematical culture with all its trials and tribulations. It shows not only differences, but similarities with other mathematical education cultures elsewhere, some of which have been presented in this issue. The influence of US and UK mathematics educators since Indonesia's independence, was deeply coloured by what became known as the 'new math', bringing with itself the same problems that occurred in the societies from which it originated. But as with other types of viruses, so this 'new math' virus seems to have been more deadly in the new career, and how Indonesia dealt with it is both to be admired and learnt from.

Atsumi Ueda, **Takuya Baba** and **Taketo Matsuura** contributed with describing the values in Japanese mathematics education from the perspective of open-ended approach. Japanese mathematics education is world-renowned for the development of its open-ended approach. This paper gives an insight into what it actually means and how it got developed – with Japanese mathematics being prominent for its success and individuality this is an extremely valuable lesson to learn.

Karmelita Pjanić follows this article with the one on the 'Origins and Products of Japanese Lesson Study'. Again, this practice that originated in Japan, has been introduced and followed throughout the world. I myself use it in my training and development of teachers and find that its benefits are innumerable. To learn about its origin is therefore enlightening and also offers invaluable information that sheds more light on the process of lesson study cycle.

‘Mathematical And Cultural Messages From The Period Between The Two World Wars: Elin Pelin’s Story Problems’ is an article by **Iordanka Gortcheva** which gives a lovely description and analysis of the mathematical problems for the classroom developed in Bulgaria by Elin Pelin, an amateur mathematician. In contrast to this, the chapter by **Aleksandar M. Nikolić** on Judita Cofman, gives an additional insight into the culture of learning mathematics in the neighbouring Serbia, during the 20th century, and her efforts in developing support structures for the young mathematicians of the region.

Last, but by no means least, was the contribution by **Maria Rosa Massa Esteve** who wrote for us about the ‘Historical activities in the mathematics classroom: Tartaglia’s Nova Scientia (1537)’. An excellent, beautifully illustrated paper, it gives very practical ideas for the use of original sources in the classrooms. This in itself is a skill that is worth learning, but at the end of this issue, this paper is also a call for all our current and future contributors to look for inspiration from their local, national, and regional histories to create sources for the classroom internationally.

We hope that we will be able to put an issue perhaps based on such future work, in a couple of years time, and see some of our current contributors submit their work again. Olivera Djokić[1] and I very much enjoyed working with you all and thank you very much for your contributions! We hope to see you all at some mathematics education event.

Snezana Lawrence[2], PhD, Guest Editor

1 olivera.djokic@uf.bg.ac.rs

2 s.lawrence2@bathspa.ac.uk

Уводна реч гостујућег уредника

Историја математике и њена примена у математичком образовању одавно су препознате као важне, и као такве иза себе имају велики број написаних и објављених радова. У скорије време (прецизније: током протекле деценије), историја математике и њена примена у математичком образовању класификоване су – како у погледу примене у учионици, тако и у погледу историје математике у математичком образовању и оспособљавању будућих наставника. Различита међународна тела посвећују пажњу историји математике и математичког образовања. Наведимо их.

1. Међународни конгрес математичког образовања (International Congress of Mathematics Education, ICME), који поседује две редовне секције у корпусу својих секција – једну посвећену историји математике у образовању и другу посвећену историји математичког образовања.
2. Историја и педагогија математике (History and Pedagogy of Mathematics, HPM) представља светско удружење академика које одржава научне скупове једном у две године; једном у четири године скуп се одржава заједно са Међународним конгресом математичког образовања, а у оним годинама када он није заједнички (две године пре/после заједничког), одржава се Европски летњи универзитет (European Summer University). Историја и педагогија математике званично је сарадник Међународне уније за математику (International Mathematics Union), чији рад помаже.
3. Национална академска удружења, као што су Канадско друштво за историју математике (Canadian Society for the History of Mathematics) и Британско друштво за историју математике (British Society for the History of Mathematics), одржавају редовне (понекад и заједничке) скупове који повезују колеге преко Атлантика.

Сва ова тела својим радом дају велики допринос у виду публикованих издања, као што су зборници са научних скупова, те три научна часописа за истраживаче са енглеског говорног подручја у чијим радовима се обрађују питања од интереса за нас: *Историја математике* (*Historia Mathematica*), *Часопис британског друштва за историју математике* (*Journal of the British Society for the History of Mathematics, BSHM Bulletin*) и *Међународни часопис за историју математичког образовања* (*International Journal for the History of Mathematics Education*). Можемо да поставимо питање – откуд онда потреба за још једним издањем као што је посебно издање часописа *Иновације у настави* (*Teaching Innovations*), које стоји пред нама?

Навешћемо два разлога.

Промена идентитета која се појавила након пада Берлинског зида односи се и на промену утицаја и начина комуникације образовних политика. Историја математике и историја математичког образовања, мада се првенствено баве самом математиком, ипак су обојене националним и међународним друштвеним контекстима из којих настају, а наш *Темат* неким од својих радова сведочи у прилог томе – то можемо узети за први разлог.

Други разлог је само учење – учење које се одвија у нашим учионицама, учење будућих наставника који се обучавају за посао у њима, као и наше сопствено учење. Овде је кључно последње – учење аутора *Темата*, али и свих чланова друштва које учи, а у фокусу наших напора лежи следеће: овим *Тематом* сагледавамо праксе и традиције друштава које обухватају међународну заједницу математичког образовања од Јапана и Индонезије, преко великих царстава, до данашње Каталоније и Србије. Такође, сам избор радова у *Темату* показује интересовање аутора који су били спремни да се укључе у процес преношења својих искустава и знања преко првог енглеског издања часописа *Иновације у настави* (*Teaching Innovations*) посвећеног математичком образовању.

Даћемо кратак опис сваког рада и искористити прилику да захвалимо ауторима на њиховом труду и професионализму који су показали док је *Темат* настајао, на одговорном приступу изузетно занимљивим проблемима којима се у радовима баве, као и на подршци коју су нам пружили да *Темат*, као нова иницијатива, угледа светлост дана.

Први чланак у *Темату* носи наслов „Историја математике и настава математике" **Мирка Дејића** и **Александре М. Михаловић**. Он доноси занимљиве квантитативне и квалитативне податке прикупљене испитивањем мишљења наставника о употреби историје математике у учионици. Реч је о одличном уводу у истраживање математичког образовања које показује везу између компетенција наставника математике и њихове свести о примени историје математике у настави.

Чланак „О култури креативности у математичком образовању" **Бронислава Чарнохе** даје задивљујући предлог како наставници могу да подрже креативност својих ученика – не само описом како би се то могло извести у учионици већ формулисањем скупа принципа који могу да се користе у настави у којој наставници подстичу развој креативности уопште. Чарноха приказује неке од скоро представљених резултата истраживања на 38. састанку међународне групе за психологију математичког образовања (PME) у 2014. години (15–20. јула), преиспитујући их на научној основи.

Чланак који даје преглед математичког образовања **на Балкану** написала сам на основу истраживања које сам извела пре неколико година и који је објављен као поглавље о математици на Балкану у *Оксфордском приручнику за историју математике* (*The Oxford Handbook of the History of Mathematics*). Њиме се остварује увид у историјске оквире балканских народа, који су развијали своје образовне праксе. Тако се и грчка и отоманска математика, по мишљењу савремених математичара, виде као две велике математичке традиције на овом простору које су снажно утицале на математичко образовање балканских народа, а које су у 19. веку зависиле од западних европских земаља. Посебно математичко образовање на Балкану створило је у Србији јаку математичку културу унутар једне школе и једног математичког генија који је испољио интелигентан и еластичан дух.

Чланак аутора **Маилизара Маилизара**, **Манахел Алафелек** и **Лиангуо Фана** даје нам историјски преглед реформе наставног програма математике и његов развој у савременој Индонезији. Један од највећих образовних система у свету, али мало познат ван земље, посебно у Европи, даје занимљив увид у математичку културу југоисточне Азије, са свим својим преиспитивањима и потешкоћама. Он показује не само разлике већ и сличности са другим културама математичког образовања, од којих су неке представљене у овом *Темату*. Све од утицаја математичких педагога из Сједињених Америчких Држава и Велике Британије до проглашења независности Индонезије било је дубоко обојено оним што је постало познато као „нова математика“, која је собом носила исте проблеме који су се дешавали у друштвима из којих су проистекли. Али као и код других вируса, тако је и овај вирус „нова математика“ смртоноснији у новом обличју; начин на који се Индонезија борила са њим јесте за дивљење и из те борбе се може доста научити о томе како се са таквим проблемом треба носити.

Ацуми Уеда, **Такуја Баба** и **Такето Мацура** дали су свој допринос *Темату* анализирајући вредности јапанског математичког образовања из перспективе отвореног приступа. Јапанско математичко образовање светски је признато као покретач развоја овог приступа. Чланак даје увид у то шта заправо значи поменути отворени приступ и како се он развијао – јапанска математика у великој мери је заслужна за његов успех и његове особености, о чему сведочи овај рад.

Кармелита Пјанић претходни чланак прати својим „Порекло и производ јапанске ’студије часа’“. Пракса поникла у Јапану описана је и праћена широм света. Користила сам је у обуци и усавршавању својих студената, будућих наставника, и сматрам да су њене предности вишеструке. Уколико желите да сазнате више о пореклу јапанске ’студије часа’, Кармелитин рад даје непроцењиве информације које осветљавају сам циклус ’студије часа’.

„Математичке и културне поруке из периода између два светска рата: проблеми текстуалних задатака Елина Пелина“ чланак је **Јорданке Гочеве** који даје добар опис и лепу анализу математичких проблема развијених у Бугарској за учионицу Елина Пелина, математичара аматера. Насупрот томе, текст **Александра М. Николића** о Јудити Цофман пружа додатни увид у културу учења математике у суседној Србији у 20. веку, и бележи њене напоре у вези са развојем структурне подршке младим математичарима у региону.

Последњи, али не и најмање важан, јесте чланак **Марије Розе Масе Естеве,** која је написала за нас рад наслова „Историјске активности на часовима математике: Тартаљина *Nova Scientia* (1537).“ Одличан, лепо илустрован рад, који нуди практичне идеје за коришћење оригиналних историјских извора у учионици. Реч је о вештини која је сама по себи вредна учења, али на крају овог издања рад Марије Розе такође позива све наше садашње и будуће ауторе да нађу инспирацију у својој локалној, националној, и регионалној историји, као и да створе изворе за учионичку праксу, који могу да постану предметом међународних истраживања математичког образовања.

Надамо се да је овај специјални број часописа *Иновације у настави* (*Teaching Innovations*) отворио поље за будућу сарадњу са међународним истраживачима у области математичког образовања. Такође, верујемо да ћемо за неколико година видети неке од наших сарадника са њиховим новим радовима, те да ћемо моћи да се вратимо истим или сличним питањима. Оливера Ђокић[1] и ја смо веома уживале у раду са вама и стога још једном желимо да вам захвалимо на драгоценом доприносу! Надамо се да ћемо вас све видети и на неком од предстојећих догађаја који су посвећени математичком образовању.

др Снежана Лоренс[2], гостујући уредник

1 olivera.djokic@uf.bg.ac.rs

2 s.lawrence2@bathspa.ac.uk

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Survey Paper

Mirko Dejić[1], PhD
Teacher Education Faculty, University of Belgrade, Belgrade, Serbia

Aleksandra M. Mihajlović, PhD
Faculty of Pedagogical Sciences, University of Kragujevac, Jagodina, Serbia

History of Mathematics and Teaching Mathematics

***Abstract**: The paper discusses the possibilities of using contents of history of mathematics as a supporting strategy in the teaching of mathematics. There is plenty of research that promotes using historical content in mathematics lessons, but only a few of them are of empirical nature. We will give the brief overview of some studies and consider different possibilities of integrating contents of history of mathematics into the teaching and learning process. Moreover, we will point out some benefits of using the history of mathematics such as: increasing students' motivation, decreasing anxiety related to the subject, building positive attitude towards mathematics, better understanding and development of mathematical concepts, changing the students' perception about mathematics, development of multicultural approach to the subject, more chances for individual work and learning by discovery, helping students to understand the role and importance of mathematics in society etc. Furthermore, we are analyzing the current state of mathematical education in Serbia and some other countries from the aspect of integrating the contents of the history of mathematics into teaching. The main goal of this paper is to investigate the teachers' beliefs and attitudes about possibilities of using history of mathematics in their practice. Based on the results of the inquiry we will suggest possible ways of how to include and use the history of mathematics in mathematics classrooms.*

***Key words**: history of mathematics, teaching mathematics, beliefs and attitudes of teachers.*

Significance of History of Mathematics

"Mathematics is one of the oldest of sciences; it is also one of the most active; for its strength is the vigour of perpetual youth."

Andrew Russell Forsyth (1858-1942)

Mathematics is a human creation, which has been developing for more than four thousand years. It emerged as a response to different social and economic needs of civilizations such as Babylon, Egyptian, Indian, Chinese, Greek, Roman, to name but a few. In earlier civilizations, the solution to mathematical types of problems lied in empirical research, whereas in later periods deductive theoretical meth-

1 mirko.dejic@gmail.com

ods were applied (Karaduman, 2010). Historical development of mathematics stresses that mathematics as a science has always been connected to economic and social context and development of society. Modern society is more than ever dependent upon technological changes and phases of its development cannot be imagined without mathematics. If we look at the development of other sciences such as physics, chemistry or biology, we may notice that mathematics played an important role in each of them. Thus, we can say that, as understanding of the world is based on scientific theories, mathematics represents an important part of human cultural and scientific heritage.

Some scientists recognize only the cultural side of studying history of mathematics (Gnedenko, 1963). They do not recognize other benefits which we will discuss shortly, and place the history of mathematics in the historical science. In their opinion, new knowledge and ideas do not rely on the past, the past can only prevent progress, and many theories are out of date. This view of the history of mathematics states that a progress comes only with new ideas which did not exist in the past, and that the study of the past is not necessary in the study of mathematics.

If we open a historical book or textbook used in educational settings in Serbia, we shall rarely find anything about a mathematician or historical description of a mathematics discovery. One may generalise that, on the grand scale, in the history of philosophy, only mathematicians of the ancient world and their works are described in any depth. Fortunately, many great mathematicians understood the necessity of studying the history of mathematics (ibidem).

Our belief is that it is necessary for a man to know what provoked the development of mathematical ideas, which methods of study was used in the past and how the problems that were posed were solved. Answers to these issues do not have only cultural and historical significance, but are important for the development of contemporary science. Characteristics of mathematics of a certain epoch, but also of contemporary mathematics, can be understood only in the context of mathematical achievements of the past. One example is how the fifth Euclid's postulate paved the path to the new non-Euclid's geometries, and they in turn formed the foundation for creating more abstract mathematical constructions and axiomatic-deduction systems. A few other examples about the nature of mathematics: in Ancient Greece mathematics was a science about spatial and quantitative relations, but today structures are dominant and the subject of the research is far wider. The term limit and function have been previously connected to mathematical analysis, but today these concepts have exceeded this application. But both measure theory and integration have their deep roots in ancient mathematics. What we are trying to say is that, to know mathematics, one needs to know its history; as Newton expressed metaphorically he had seen further than others by standing upon the shoulders of giants.

It seems therefore that to learn mathematics, it is significant to follow the historical changes in mathematics. Besides knowing historical path of ideas, concepts and facts, which help us to form methodological path, we can create the basis for better understanding of contemporary concepts and views in mathematics, which will modernize methodical directions, which have been frequently outdated.

In the teaching process, special attention should be given to developing positive attitudes of students towards mathematics (Ma&Kishor, 1997, Akinsola&Olowojaiye, 2008, Memnun&Akkaya, 2012). One of the ways to achieve this is to show and convince the students that mathematical knowledge can make their life easier and improve it. But most importantly, a common sense tells us that Mathematics teaching should be organized in the environment in which students will eagerly acquire new knowledge by their own intellectual efforts and abilities. One of the pedagogical tools for achieving these goals is history of Mathematics, and we will now look at how we use this tool in our work.

Significance and role of History of Mathematics in teaching

One of the reasons we use history of mathematics in the teaching and learning of the subject is that we believe that if mathematical theories are seen only through their final formulation, without historical interpretations, students can gain a wrong impression about mathematics: they seem to then see it as an artificial creation, which serves mental imagination, but has no connection to practical work or real-life contexts. This can be overcome when students, through historical facts, understand that mathematics from its foundation up to now has played one of the most significant roles in all areas of human life. Students can gain an insight into mathematical concepts in a deeper and more interesting way and from many examples from the past can understand that mathematics is not an isolated discipline (Carter, 2006 according to Goktepe, Ozdemir, 2013).

The idea of using history of mathematics in mathematical education is not new. More than a century ago, Zeuthen (Furinghetti, Radford, 2002) wrote a book on the history of mathematics aimed for teachers. Zeuthen considered history of mathematics to be an integral part of general education of teachers. Almost at the same time, in 1894, Florian Cajori noticed in history of mathematics an inspirational source of information for teachers (Karaduman, 2010). Freudenthal (1981) thought that introducing the history of mathematics into the education of mathematics teachers would provide a background to their mathematical knowledge (Lawrence, 2009). As a starting point towards more serious scientific studies, we can determine the foundation of the working group for History and Pedagogy of Mathematics in 1972 at the Second International Congress on Mathematical Education (ICME), and the foundation of the International Study Group on Relations between History and Pedagogy of Mathematics in 1976 (Furinghetti, 2005). In the last 20 years the awareness of important role and application of history of mathematics in the process of teaching and learning has been increasing (Goktepe, Ozdemir, 2013). *Mathematical Association of America founded* The Institute about the History of Mathematics and Its Use in Teaching (IHTM) in 1995. At the meeting of the International Congress on Mathematics Education (ICME) in 1996, the significance of the history of mathematics in motivation of students and using mathematics in teaching activities was stressed. Moreover, at the International Teaching Mathematics Conference (ICTM-2) in 2002 a special panel section was organized with the title The Role of the History of Mathematics in Mathematics Education.

Integrating history of mathematics into the teaching practice helps students understand that mathematics is not fixed and final system of knowledge, but that it represents live developmental process, which is closely linked to other branches of science (Karaduman, 2010). In pedagogical sense, students form a scientific view of the world and become aware of the fact that mathematics always has an important role in the development of entire culture of a certain epoch. Through genesis of a certain concept, students realize that mathematical truths are understood or discovered through usually a very long and hard work. History of mathematics helps students understand that errors, doubts, intuitive reasoning, discussions and alternative approaches are not only legitimate, but an integral part of mathematics in the making (Tzanakis & Arcavi, 2000, Paschos T. et al., 2004). History of Mathematics represents an inseparable part of mathematics (Goktepe, Ozdemir, 2013).

There are many studies that promote using history in mathematics classes, and they point at the advantages it brings. Wilson & Chauvot (2000) talk about four main benefits of using history of mathematics in the classroom. According to them, its integration into teaching sharpens problem solving skills, makes the basis for better understanding the contents, helps students make different mathematical connections and enlightens the connection between mathematics and society (according to Burns B., 2010). Bidwell (1993) points out that history of mathematics gives mathematics human dimension (according to Kaye E., 2008). Marshall & Rich (2000) stress that history of mathematics enriches mathematical curriculum and demystifies mathematics, showing that it is the human creation (according to Roscoe, 2010). Jankvist (ibidem) talks about other advantages of using history of mathematics through increased motivation (through creating interest for the subject), and decreased intimidation (through understanding that mathematics is a human creation and that its creators also had to make a great effort in order to come to cognition). According to him, the use of history of mathematics can show students new perspectives on the discipline and enable them to have better insight into specific mathematical contents. On the other hand, the history of Mathematics can serve the teacher as a guide through difficulties, which students face when learning a certain mathematical topic. Those difficulties are often similar to those which were encountered through some historical development of certain concepts.

Gnedenko (Gnedenko, 1996, p. 132) mentions following reasons of why history of mathematics should be studied:

1. History of mathematics gives us a wide perspective of development of mathematics itself, development of its concepts and problems, connection to the praxis, tendencies for generalization and proving scientific assumptions

2. History of mathematics is a part of general history which tells us how mankind was made to develop mathematics and to use its results

3. History of mathematics is one of the prerequisites for further development of contemporary mathematics

4. It is the basis of scientific methodology and one of the most significant sources of the analysis of cognitive processes

5. History of mathematics contributes to improving mathematics teaching

6. History of mathematics is integral part of general human culture.

Many studies, by their results, support the fact that integration of history of mathematics in classes influences students' achievements, their interests and attitudes. Marshall (2000) noticed that using history of mathematics in classes had a positive effect on attitudes of students of secondary schools towards learning mathematics (Goktepe, Ozdemir, 2013). İdikut (2007) made an experimental research with the seventh grade students (ibidem). The main aim of this research was to examine the effect of using history of mathematics as a pedagogical tool on attitudes and achievements of students. The results have shown that there has been no effect on students' attitudes and the level of cognition, but that there was a positive effect on students' achievement in mathematics classes. Karaduman (2010) performed an experimental research in parallel groups with 90 students of primary school (age 10, 11 and 12); students of the experimental group received differentiated instruction and used teaching material with contents from the history of mathematics. Post-test results showed that experimental programme had positive effect on students' achievement, i.e. that students of the experimental group had statistically better results than the students of the control group.

Considering the fact that a teacher is the one who plans, prepares and performs mathematics teaching, his/her role must not be neglected in the teaching process. To which extent and in which ways the history of mathematics will be integrated into teaching depend on attitudes and readiness of teachers. Ho (2008) conducted survey among teachers of Singapure schools. According to the results, the author points out three, according to him, significant aspects of using history of mathematics in teaching: potentials, limitations and risks. According to him, most of the teachers do not recognize potentials for using historical approach (contribution to better understanding of the topic, creating a more favourable environment for learning, developing more positive attitudes of both teachers and students). Main limitations, according to teachers, are lack of teacher training in this respect, the lack of time in the curriculum, and the difficulties in assessing the knowledge of students. As to the risks inherent in using history in the teaching of mathematics, most teachers point out they are worried to overstress historical content in comparison to the mathematical, and their possible inability to create mathematical connections in a short time that lesson gives to presentation of such ideas.

In this paper, we wanted to examine the current state of affairs in relation to the use of history of mathematics in primary schools in Serbia, formed by teachers' perceptions.

Research Methodology

The main research interest of this paper was to examine the current state in lower grades of the primary schools in Serbia concerning the use of the history of mathematics in teaching as well as the attitudes and willingness of teachers to include such content into their work. The aim was realized through the following research tasks:

- To determine to what extent teachers had the opportunity during their schooling to learn some history of mathematics
- To determine to what extent teachers in their current work use the history of mathematics
- To examine what sources teachers use to learn about the history of mathematics, and what sources they use in the classrooms?
- To investigate the readiness of teachers to introduce into their work the history of mathematics
- Researching beliefs of teachers about the role, significance and possibilities of using history of mathematics in teaching.

The general hypothesis of the research is that teachers in elementary mathematics teaching do not use contents of history of mathematics to any great extent.

The survey technique was used in the empirical research, and a questionnaire was created to examine the beliefs and attitudes of teachers about application of the history of mathematics in the teaching of the subject. The questionnaire consisted of two parts. The first part included demographic characteristics of a chosen sample (such as the geographical location of the school in which teachers work, their work experience and level of education). The second part consisted of 9 questions, i.e. 7 questions of the closed type and two five-point Likert scales (the first scale had six and the second one nine items). Data collected by the questionnaire were analysed quantitatively. The statistical analyses included methods of descriptive statistics (frequency, percentage, mean, standard deviation, coefficient of variation) and χ^2 test. The independent variables in the data analysis were the level (degree) of education[2], working experience, and the geographical location of the school.

The research was conducted during the school year 2012/2013 and included the sample of 112 teachers from five areas of the Republic of Serbia

2 Before 1993 primary teachers in Serbia studied at the Teacher Training Colleges (post-secondary schools). In 1993 the Assembly of the Republic of Serbia adopted law on establishing teacher training faculties. According to this classification teachers who graduated from colleges have 6th level of education, and teachers who graduated from faculties have 7th level of education.

(Jagodina, Rekovac, Kragujevac, Beograd, Vršac). Structure of the sample, according to the degree of education was given in the table 1.

Table 1. Structure of the sample in relation to the degree of education

	6th level	7th level	Total
Number of teachers	80	32	112
Percentage (%)	24.1	75.9	100

Structure of samples in relation to the geographical location of schools in which interviewees work was given in table 2.

Table 2. Structure of the sample in relation to the geographical location of the school

	Urban	Rural	Total
Number of teachers	27	85	112
Percentage (%)	71.4	28.6	100

Structure of samples in relation to working experience is given in the table 3.

Table 3. Structure of the sample in relation to the working experience

	Numbers of years of working experience				
	0-10	11-20	21-30	31and more	Total
Frequency	26	33	38	15	112
Percentage (%)	23.2	29.5	33.9	13.4	100.0

The data analysis of the research is specially going to be focused on replies in which statistically significant differences were shown, according to independent variables.

Results of the research

1. The first task of the research was to examine whether teachers during their schooling had the chance to get to know enough of the history of Mathematics so that could meaningfully employ it in their teaching. Only 6.3% teachers responded to have met with such content in their primary education, although a greater percentage replied that they had the chance to get to know some of the history of mathematics during their secondary education, 26.8% to be precise. 69.6% of the total number of teachers interviewed had experience of the history of mathematics during their university education.

Through this analysis, we could see that there is statistically a significant difference in relation to the education of teachers (χ^2=5,326, df=1, p=0,021), and this is shown in the table below:

Table 4. Distribution of replies of teachers in relation to education

			I had the opportunity to get to know contents of history of Mathematics during university or college studies	
			no	yes
Level of education	6th level (undergraduate)	F	13	14
		%	48.1%	51.9%
	7th level (postgraduate)	F	21	64
		%	24.7%	75.3%
Total		F	34	78
		%	30.4%	69.6%

There is a greater percentage of teachers with the 7th level of education in comparison to those with the 6th level, who had opportunity to get to know the contents of history of mathematics during studies. Low value of the phi coefficient (Φ=0.218, p=0.021) indicates that there is low association between the level of education of teachers and experience of the history of mathematics during their university education.

2. Almost a fifth of the total number of teachers 19.8% replied that they used almost always or frequently some content from the history of mathematics. 59.5% of the total number use them rarely, whereas one fifth, 20.1% does not use them at all. We compared this with the previous results – whether teachers had opportunity to learn about the history of mathematics in their own education.

Teachers who learnt some history of mathematics during their own studies use it all the time or frequently in their teaching work in higher percentage (22, 1%), in comparison to those who did not have such experience (14.7%). However, it has been shown that there is no statistically significant difference (χ^2=4.198, df=2, p=0.123).

We found that there is statistically significant difference when it comes to the geographical location of the school (χ^2=14.590, df=2, p=0.001). There are a greater number of teachers of urban schools who use the history of mathematics in their work (21.5%), compared to 15.6% of teachers in rural schools (table 5). In addition, only 11.4% of teachers of urban schools replied that they almost never use history, whereas the percentage of rural schools' teachers were greater (43.8%). Cramer's V coefficient (V=0.363, p=0.01) suggests a moderate correlation between the geographical location of the school and the use of history mathematics in teaching. Possible reasons might be that rural schools libraries are usually poorly equipped comparing to urban schools' libraries. Furthermore internet is still less available in some rural areas.

Table 5. Distribution of teachers' replies in relation to the geographical location of the school

<table>
<tr><td colspan="3" rowspan="2"></td><td colspan="3">In my work, I use History of Mathematics contents</td></tr>
<tr><td>almost always and often</td><td>rarely</td><td>almost never</td></tr>
<tr><td rowspan="4">Geographical location of the school</td><td rowspan="2">Urban</td><td>F</td><td>17</td><td>53</td><td>9</td></tr>
<tr><td>%</td><td>21.5%</td><td>67.1%</td><td>11.4%</td></tr>
<tr><td rowspan="2">Rural</td><td>F</td><td>5</td><td>13</td><td>14</td></tr>
<tr><td>%</td><td>15.6%</td><td>40.6%</td><td>43.8%</td></tr>
<tr><td colspan="2" rowspan="2">Total</td><td>F</td><td>22</td><td>66</td><td>23</td></tr>
<tr><td>%</td><td>19.8%</td><td>59.5%</td><td>20.7%</td></tr>
</table>

It is interesting to note that the history of mathematics content is used by teachers who have greater working experience (more than 20 years of working experience) 26.9%, whereas this is done by 13.6% of those who work less than 20 years. Nevertheless, this difference has not been shown to be statistically significant (χ^2=3.106, df=1, p=0.078).

3. We wanted to examine which sources of information teachers use to find appropriate contents from the history of mathematics. Replies of teachers indicate that the greatest percentage of teachers finds such content by using secondary literature 50.9%, 37.5% from the Internet, and 19.6% from television. It has been shown that there is statistically significant difference (χ^2=7.812, df=1, p=0.005) concerning the level of education of teachers (table 6). 44.7% of teachers with the level seven use the Internet for finding historical content, whilst only 14.8% of teachers with level six do this. However, low value of the phi coefficient (Φ=0.264, p=0.005) shows that there is low association between the level of education of teachers and using internet for finding contents from the history of mathematics.

Table 6. Distribution of teachers' replies according to the level of education

<table>
<tr><td colspan="3" rowspan="2"></td><td colspan="2">I find contents from the history of Mathematics by using the Internet</td></tr>
<tr><td>no</td><td>yes</td></tr>
<tr><td rowspan="4">Level of education</td><td rowspan="2">6th level</td><td>F</td><td>23</td><td>4</td></tr>
<tr><td>%</td><td>85.2%</td><td>14.8%</td></tr>
<tr><td rowspan="2">7th level</td><td>F</td><td>47</td><td>38</td></tr>
<tr><td>%</td><td>55.3%</td><td>44.7%</td></tr>
<tr><td colspan="2" rowspan="2">Total</td><td>F</td><td>70</td><td>42</td></tr>
<tr><td>%</td><td>62.5%</td><td>37.5%</td></tr>
</table>

4. We asked the teachers what kind of contents and activities from the history of Mathematics they use in their classes. Distribution of replies was shown in table 7.

Table 7. Distribution of teachers' replies to question 4

In my classes I use:	f	%
Stories about famous mathematicians	41	36.6
Anecdotes	38	33.9
History of development and origin of some mathematical symbols	35	31.3
History of development and origin of some mathematical concepts	26	23.2
Something else	6	5.4

As in table 7, the greatest percentage of teachers use stories about famous mathematicians and anecdotes from the history of mathematics. Teachers from urban schools use stories from life of the well-known mathematicians around 42.5%, in comparison to those from the rural schools 21.9% (χ^2=4,190, df=1, p=0,041, C=0,190). Distribution of the teachers' replies is seen in table 8.

Table 8. Distribution of teachers' replies in relation to the geographical location of the school

			I use stories about famous mathematicians in my work	
			no	yes
Geographical location of the school	urban	F	46	34
		%	57.5	42.5
	rural	F	25	7
		%	78.1	21.9
Total		F	71	41
		%	63.4	36.6

The situation is similar when it comes to using anecdotes (χ^2=4,604, df=1, p=0,032, C=0,199). 40% of teachers of urban schools use the anecdotes, and only 18.8% of teachers of rural schools do this (table 9). Since most teachers use literature as a main source of information, the reason might be that libraries of the rural schools are less equipped than libraries of the urban schools.

Table 9. Distribution of teachers' replies in relation to the surrounding in which the school is situated

			I use anecdotes in my work	
			no	yes
Geographical location of the school	urban	F	48	32
		%	60.0	40.0
	rural	F	26	6
		%	81.3	18.8
Total		F	74	38
		%	66.1	33.9

It has been shown that there is a greater percentage of teachers with university education 41.2% who use anecdotes in comparison to teachers with college education 11.1% (χ^2=8,263, df=1, p=0,004, C=0,262). Comparing these results with the results from the previous questions, we can say that one of the possible reasons can be that teachers, who completed university studies, had learnt more of the history of mathematics than those who graduated at the college.

5. Replying on the question whether the "History of Mathematics contents should be included into the teaching curriculum of Mathematics in primary schools", most of the teachers had positive reply: 48.1%, whereas 32.4% was not sure and 19.4% responded negatively. This question was affirmatively answered by a greater percentage of teachers who have been working less than 20 years at schools (54.4%) in comparison to those who have been working more than 20 years (41.5%). However, this difference has not been shown to be statistically significant (χ^2=5.310, df=2, p=0.070). In addition, there is a higher percentage of teachers with university education (51.9%) who have affirmative replies in comparison to those with college education (37%). Nevertheless, testing teachers' replies according to this independent variable, there was no statistically significant difference (χ^2=2.853, df=2, p=0.240).

6. When asked whether they were ready to include contents from the history of mathematics into their work to a greater extent, most of the teachers gave positive replies 79.3% (13.5% responds that that they are ready to include these contents, and 65.8% reply that they are ready to include them occasionally). There is no significant difference according to any independent variable: concerning the level of education of teachers (χ^2=1.523, df=3, p=0.677), concerning the geographical location of school (χ^2=6.748, df=3, p=0.080), concerning the working experience (χ^2=1.516, df=3, p=0.679).

7. The main aim of this question was to examine the attitude of teachers about the significance of using contents of history of mathematics in teaching. Table 10 presents each of the item codes used in the first Likert's scale.

Table 10. Definition of the Likert's scale item codes

Code	Items
C1	Contents from the history of Mathematics offer teachers opportunities to motivate students
C2	Contents from the history of Mathematics offer teachers opportunities to provoke curiosity
C3	Contents from the history of Mathematics offer teachers opportunities towiden students' knowledge
C4	Contents from the history of Mathematics offer teachers opportunities to make connections between Mathematics and everyday life
C5	Contents from the history of Mathematics offer teachers opportunities to connect Mathematics and other fields and teaching subjects
C6	Contents from the history of Mathematics offer teachers opportunities to make some mathematical contents closer to students

Results show that, in general, the attitudes of teachers in regards to the possibilities to use the history of mathematics in mathematics teaching were positive and homogeneous, and this is encouraging. The overall average for the Means of teachers' attitudes toward the possibilities to use history contents was M=4.12 and standard deviation SD = 0.88. Coefficient of variation value (C_V= 21.40) indicates that teachers' attitude towards possibilities to use history contents in mathematics teaching is relatively the same.

Most of the teachers believe that contents from the history of mathematics help their students to understand better some mathematical contents, enable them to expand their knowledge, provoke their curiosity and give them opportunity to connect mathematical contents with learning and teaching of other subjects and everyday life. In addition, most of the teachers responded that contents from the history of mathematics help them to motivate students (this statement is approved totally or partially by 72.4% teachers, but there is a number of those who are not sure 8.9% and who disagree

Table 11. Distribution of teachers' replies to question 7

Item Codes	N	I totally disagree	I disagree	I am not certain	I partially agree	I totally agree	Mean (M)	Standard Deviation (SD)	Coefficient of Variation (C_V)
C1	112	2	10	10	49	32	3.96	0.99	24.98
		1.8%	8.9%	8.9%	43.8%	28.6%			
C2	112	3	8	2	44	1	4.22	1.00	23.65
		2.7%	7.1%	1.8%	39.3%	5.5%			
C3	112	0	6	9	54	8	4.16	0.80	19.30
		0%	5.4%	8.0%	48.2%	3.9%			
C4	112	1	7	10	51	6	4.09	0.89	21.76
		0.9%	6.3%	8.9%	45.5%	2.1%			
C5	112	0	4	12	59	3	4.12	0.75	18.09
		0%	3.6%	10.7%	52.7%	9.5%			
C6	112	0	5	15	44	4	4.18	0.84	20.14
		0%	4.5%	13.4%	39.3%	9.3%			

10.7%). It has not been found whether the attitudes about the advantages and importance of using the history of mathematics in teaching were determined by years of working experience, educational degree, and geographical location of the school.

8. One more issue we have dealt with in this research was to investigate causes that affect the use of the contents of history of mathematics in teaching. When creating Likert's scale, we used some items that Siu (2007) used in his research. Siu examined opinion of mathematics teachers about the possible reasons for not using the history of mathematics in the classroom. He created the list of 16 unfavourable factors. We adjusted eight items, from the instrument which was used by Siu, for our scale aimed at primary teachers.[3] Table 12 presents each of the item codes used in the Likert's scale.

Table 12. Definition of the Likert's scale item codes

Code	Items
HM1	I do not use contents from history of Mathematics in my classes, because there is not enough literature about their application in classes
HM2	I do not have enough time to use contents from history of Mathematics because of the compulsory curriculum
HM3	Students do not have enough general knowledge, so that they can understand and appreciate contents from the history of Mathematic
HM4	Using contents from the history of Mathematics does not influence better students' achievement
HM5	I am not sufficiently trained for using contents of the history of Mathematics in teaching
HM6	Students do not like contents from the history of Mathematics
HM7	Contents from the history of Mathematics are boring for students
HM8	Using contents from the history of Mathematics can only confuse students
HM9	Contents from the history of Mathematics do not have connection to Mathematics

3 Items 1, 2, 3, 4, 5, 6, 7 and 9 (table 11) are modified items from the questionnaire used by Siu (2007).

Values of the calculated statistical parameters (mean, standard deviation, coefficient of variation) show that teachers believe that the greatest obstacle for using history of mathematics in teaching are the insufficient use of appropriate literature and lack of time (table 13). Nevertheless, values of the coefficient of variation show that attitudes of teachers on these issues are heterogeneous; 54.5% of teachers think they haven't got enough time for historical research, but more than half of the teachers, 50.9%, agree with the statement that there is not enough literature, whilst the second half of our population is not sure or disagrees. Siu (2007) got similar results: 53% of mathematics teachers see the problem in insufficient time in the classes, and 50% think that the problem is the lack of literature and adequate teaching materials. It is interesting that 46.5% of teachers disagree (totally or partially) with the statement that they are not sufficiently trained for using the history of mathematics in teaching, 19.6% are not sure, and 26.8% agree with this statement. (Siu in his research came to the result that 78% of mathematics teachers think that they lack adequate training for using historical content). In relation to this issue then, the attitudes of teachers are heterogeneous.

It is encouraging to find that, for example, the primary teachers in our survey negatively marked the statement that the history of Mathematics does not have any connection to mathematics. Yet, despite the fact that 70.6% of teachers totally or partially disagree with this statement, there is a smaller percentage of those who are not sure (14.3%), and those who think that the history of mathematics has nothing to do with mathematics make the rest: 6.3%. When it comes to the teachers of mathematics, Sui (2007) finds that this statement is not approved by 87.49%. Primary teachers are divided in their opinion about the statement that using contents from the history of mathematics does not influence pupils' achievement for the better. The greatest percentage is given to those who say they are not certain in this respect: 27.7%. Siu came to the similar results.

Table 13. Distribution of teachers' replies to question 8

Item code	N		I totally disagree	I partially disagree	I am not certain	I partially agree	I totally disagree	No answer	Mean (M)	Standard Deviation (SD)	Coefficient of Variation (C_V)
HM1	103	f	9	17	20	38	19	9	3.40	1.22	35.77
		%	8.0	15.2	17.9	33.9	17.0	8			
HM2	104	f	19	12	12	44	17	8	3.27	1.37	41.81
		%	17.0	10.7	10.7	39.3	15.2	7.1			
HM3	104	f	21	26	16	31	10	8	2.84	1.32	46.37
		%	18.8	23.2	14.3	27.7	8.9	7.1			
HM4	105	f	22	20	31	22	10	7	2.79	1.26	45.19
		%	19.6	17.9	27.7	19.6	8.9	6.3			
HM5	104	f	33	19	22	28	2	8	2.49	1.25	50.05
		%	29.5	17.0	19.6	25.0	1.8	7.1			
HM6	100	f	40	20	24	13	3	12	2.19	1.19	54.18
		%	35.7	17.9	21.4	11.6	2.7	10.7			
HM7	104	f	49	19	20	12	4	8	2.07	1.22	58.86
		%	43.8	17.0	17.9	10.7	3.6	7.1			
HM8	102	f	48	17	21	16	0	10	2.05	1.15	55.95
		%	42.9	15.2	18.8	14.3	0	8.9			
HM9	102	f	61	18	16	7	0	10	1.70	0.97	5736
		%	54.5	16.1	14.3	6.3	0	8.9			

The statement that contents from the history of mathematics can only confuse pupils in the classrooms is completely and partially disapproved by 58.1% of teachers. Similarly, most of the teachers totally or partially disagree that contents of the history of mathematics is boring for their pupils (60.8%), that pupils do not like it (53.6%), or that pupils themselves do not have enough general knowledge in order to understand and appreciate such content (42%). Considering the values of the coefficient of variation, we can conclude that attitudes of teachers are heterogeneous for these questions. Siu finds that 36% of mathematics teachers believe that students do not have enough general knowledge to appreciate historical contents.

Replies of teachers to these questions were compared to the replies on the question to which extent they use the content from the history of Mathematics in their work. The greatest percentage of teachers 57.8% who almost never or rarely use such content replied that this was because they did not have enough time in their lessons due to the need to cover the compulsory programme. 52.4% replied that they did not have access to relevant literature.

9. Considering the fact that one of the forms of compulsory professional development of teachers is attending accredited seminars, we wanted to examine whether the teachers would like some of the topics at this seminar to be devoted to possibilities of applying history of mathematics in primary

teaching. The greatest percentage of the total number of teachers replied affirmatively, 68.8%. Some teachers were not sure, 12.5% and 18.8% of them responded negatively. We compared answers to these questions to replies to the previous question.

Most of the teachers (58.3%) who said they wanted to hear more about the history of mathematics in seminars replied that the reason they do not use more of history in their lessons was the lack of literature and lack of time during regular classes. About one half of these teachers, 56.2% think that they are trained, and 15.1% are not sure, whereas 28.8% think that they are not sufficiently trained to include such content in their work.

Concluding remarks

It is very important for future primary school teachers and mathematics teachers to get to know the genesis of mathematical concepts and statements and many studies and research point at this (Schubring et all., 2000, Dejić, Egerić, 2010, Goktepe, Ozdemir, 2013). In teaching and learning of mathematics, students often form certain mathematical concepts in the way these concepts have been themselves formed: direct counting, measuring, observation, etc. of the real objects. Of course, the students do not go through the complete historical development (which sometimes lasts for centuries) in learning the certain mathematical concepts, but use shorter routes which can be facilitated by appropriate methodological transformation of the mathematical contents (Dejić, Egerić, 2010).

In mathematics teaching in the lower grades of the primary school and later, teachers can use facts from the history of mathematics to get students interested in the topic they teach. Telling appropriate anecdotes about great mathematicians, contributes to pedagogical work by example in the best possible way. There is a general agreement that famous historical anecdotes are effective in order to break monotony and make the class more interesting (Ho, 2008). If children like a mathematician, they may get interested in his/her work more. Gauss was only 9 when he managed to add numbers from 1 to 100 in a very short time. For primary pupils it can be very interesting to understand in which way he did this. Thales managed to measure the height of Cheops pyramid by using its shadow, and was known as one of the seven wise man of Greece, both of these facts can be easily used in the classroom to engage children. Various measuring in nature can lead children to forming the concept of length, area or volume, in the same way the ancient Egyptians did this living on the Nile River. For example, similarly to the ancient Egyptians, students can get the right angle with the aid of a rope with knots. Bidwell (1993) stresses that history of mathematics can be used in teaching in three ways: anecdotal display, injecting anecdotal material as the course is presented, and teaching topics in the way they have been developed through the history (Haverhals, Roscoe, 2010).

Results of our research show that whilst teachers have a positive attitude towards the history of mathematics they don't integrate it into teaching much. The most common reasons are lack of appropriate literature and methodological guidelines, and lack of time due to the delivery of the compulsory curriculum. Unfortunately, despite the positive attitude of majority of teachers who took part in this research, a certain number of them were reluctant to use historical contents in their teaching, or were not sure of the effects of such approach (some believed that they would bore their pupils, others that that pupils would not like such approach, and some that the use of history of mathematics in mathematics teaching does not influence achievements of pupils). It was encouraging, for us, to find that during their studies at the teacher training institutions, teachers reported having met with some content from the history of mathematics.

On the other hand, the history of mathematics cannot be found in our curriculum, and it is not mentioned in many textbooks.[4] Results of our research show that there is at least an interest of teachers to participate in accredited seminars which would cover the topics about the inclusion of the history of mathematics in teaching. Unfortunately, such programmes do not currently exist. Mathematical society *Archimedes* has been organising different seminars for teachers of mathematics and sometimes they present topics relating to the history of mathematics. Also, Serbian mathematical congresses, held every fourth year, have a section for the topics on teaching and the history of mathematics. So although not enough attention has so far been paid to the connection between the history of mathematics and mathematics teaching in our country, we believe that this can be improved. One of the ways is to organise seminars by professional associations in which teachers would be educated how to use history of mathematics in their teaching. Journals for students and teachers could contain topics from the history of mathematics that would follow mathematics curriculum. When writing mathematical textbooks, special attention could be paid to appropriate contents from the history of mathematics, which would expand students' knowledge in social and cultural context. Trifunović (1980) suggests writing separate textbooks from the history of mathematics as it is the case in some other countries (Russia, France, Italy, Romania).

It should be stressed that when we talk about the application of the elements of the history of mathematics in mathematics education, we do not talk about historicism. Too much historical material, which would be learnt for the sake of learning, would burden mathematical teaching. A teacher must be trained for methodological adjustment and application of the suitable contents. All of this implies that special attention should be given to the education of teachers, because the teacher is, and remains the major factor of students' achievements in the educational process.

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др Мирко Дејић
Учитељски факултет, Универзитет у Београду
др Александра Михајловић
Факултет педагошких наука, Универзитет у Крагујевцу

Историја математике и настава математике

У овом раду бавићемо се могућностима примене садржаја историје математике као подршке настави математике. Постоји велики број истраживања која промовишу коришћење историјских садржаја на часовима математике, али је мали број њих емпиријског карактера. У раду ћемо дати кратак приказ неких истраживања, указаћемо на различите могућности укључивања садржаја историје математике у наставу и истакнућемо неке предности њиховог коришћења, попут: повећања мотивације ученика, смањивања страха од предмета, хуманизације математике, изградње позитивног става према математици, бољег разумевања математике и развоја математичких појмова, промене перцепције ученика о математици, развијања мултикултуралног приступа, пружања ученицима могућности да самостално истражују, бољег схватања улоге и важности математике у друштву итд. Такође, анализираћемо ситуацију у математичком образовању у неким другим земљама и у Србији са аспекта интеграције садржаја историје математике у наставу. Главно истраживачко интересовање нашег рада односило се на утврђивање постојећег стања у нижим разредима основне школе по питању коришћења садржаја историје математике у настави, као и ставова и спремности учитеља да у већој мери у свој наставни рад укључе ове садржаје. Основни циљ рада било је испитивање мишљења и ставова учитеља о могућностима коришћења садржаја историје математике у наставном раду. Општа хипотеза спроведеног истраживања је да учитељи у почетној настави математике не користе у довољној мери садржаје историје математике. Истраживање је спроведено школске 2012/2013. године и обухватило је узорак од сто дванаест учитеља из пет општина Републике Србије. За прикупљање података коришћена је техника анкетирања. Резултати истраживања показали су да, без обзира на чињеницу да већина учитеља има позитиван став према коришћењу историје математике у настави, ипак, то у знатно мањем проценту примењују у пракси. Најчешћи разлози су недостатак одговарајућих књига и методичких упутстава, као и недовољно времена због реализације обавезног програма. Показало се да постоји интересовање учитеља да се на акредитованим семинарима више пажње посвети примени историје математике у настави. Требало би садржаје историје математике укључити и у часописе за ученике, али и за наставнике, а нарочиту пажњу треба обратити при писању математичких уџбеника.

Кључне речи: историја математике, настава математике, мишљење и ставови учитеља.

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Bronislaw Czarnocha[1]**, PhD**
City University of New York, Hostos Community College, Mathematics Department, NY, USA

Original Article

On the Culture of Creativity in Mathematics Education

Abstract: *Culture of creativity in mathematics education is grounded in definitions of creativity which underline our research and efforts of its classroom facilitation. However, the statement "there is no single, authoritative perspective or definition of creativity in mathematics education" (Mann, 2006; Sriraman, 2005; Leikin, 2011, Kattou et al., 2011) leaves practitioners without an idenifiable viewpoint in teaching. Therefore culture of creativity in mathematics education doesn't have solid foundations conflating, among other things, a research into creativity with research into giftedness. Prabhu and Czarnocha (2014) have argued at PME 38 for the acceptance of* ***bisociation*** *of Koestler's Act of Creation, that is a spontaneous leap of insight' as the authoritative definition of creativity. The paper presents this bisociation theory of an "Aha!" moment and identifies this moment as one during which mind can focus and eliminate inhibiting habits of mind. The paper explores cultural values brought forth by the new definitions of creativity such as its democratization, the unity of creativity, motivation in learning, and the simultaneity of attention. The examples and methods of classroom facilitation are henceforth presented. The distinction between bisociative and associative thinking shows and introduces the concept of simultaneity of attention as new type of attention in learning (Mason, 2008).*

Key words: *creativity, bisociation, 'Aha moment', simultaneity of attention.*

Introduction

The elementary meaning of the term 'culture' is probably about the way people do things. The Oxford English Dictionary looks upon culture as "arts and other manifestations of human intellectual achievement regarded collectively". Thus 'culture' can denote both processes of cultivation as well as their results, the objects of cultivation could encompass such activities as growing plants, customs, arts and, as is of interest to us, results of human intellectual achievements.

Of our primary interest in this paper is the definition of culture of creativity in mathematics education and the efforts in its facilitation in the classroom. We here reflect on the statement (quite possibly accepted within the profession that "there is no single, authoritative perspective or definition of creativity in mathematics education") (Mann, 2006; Sriraman, 2005; Leikin, 2011, Kattou et al., 2011).

1 bczarnocha@hostos.cuny.edu

The Wallas (1926) definition of creativity based on the Gestalt[2] theories postulates the following general process of **preparation, incubation, illumination** and **verufication**

The second definition measures the products of creativity through Torrance Tests of Creative Thinking (1974). It involves simple tests of divergent thinking and other problem-solving skills, which are scored on:

- **Fluency** – The total number of interpretable, meaningful and relevant ideas generated in response to the stimulus.
- **Originality** – The statistical rarity of the responses among the test subjects.
- **Elaboration** – The amount of detail in the responses.

Leikin (2007) and Silver (1997) transformed creativity to fluency, flexibility and originality making the definition one of the bases for understanding creativity in mathematics education. While the Wallas' definition focuses on the psychological neighborhood of the creative insight, the Torrance definition addresses the quantity and rarity of its products. Neither of these definitions focuses on the creative act itself as the spontaneous insight - the content of the 'Aha moment', or of the 'Eureka' experience. This kind of absence of focused balance in existing literature makes researchers working in the area of mathematical creativity to reflect about the absence of 'authoritative definition of creativity'. Together with that absence comes the 'looseness' of our culture of mathematical creativity, and this in itself might have negative impact upon nurturing creativity in mathematics classrooms, and beyond.

In fact, the investigation by Leikin (2009: 129-143) indicates that the design of instruction and research based on the Torrance tests of Creative Thinking actually lowers the creativity. The authors point, we believe correctly, to the fluency and flexibility as the carriers of the habit which diminished the originality of student "when students become more fluent they have less chance to be original" This complementary relationship between fluency and flexibility on one hand and creativity on the other hand, determines attitudes when conducting the research into creativity based on definition, because such approach may result in undesired lowering of creativity while impacting negatively on culture of the field.

Culture of creativity in the field corresponds to the value we attach to the creativity itself. The absence of the 'authoritative approach or definition of creativity' in mathematics education reflects the ambiguities contained in the value of creativity as valued by mathematics educators. It is time then to look for such a definition of creativity in mathematics that places its understanding on firmer, unambiguous foundation.

Bisociation

The theory developed by Arthur Koestler in his 1964 work, *Act of Creation*, gives us such definition. It builds on our understanding of creativity on the basis of a thorough inquiry into the 'Aha moment' – a bisociative leap of insight, the very site of creativity according to Sriramana (2005). Arthur Koestler defines 'bisociation' as "the spontaneous flash of insight, which…connects the previously unconnected frames of reference and makes us experience reality at several planes at once… " (Koestler, 1964: 45). Koestler clarifies the meaning of 'insight', by invoking Thorpe's 1956 definition of insight "an immediate perception of relations" (Koestler, 1964: 548). Koestler also refers to Koffka's (Koffka, 1935)

2 The idea of Gestalt has its roots in theories by Johann Wolfgang von Goethe and Ernst Mach. Max Wertheimer is to be credited as the founder of the movement of *Gestalt* psychology. The concept of Gestalt itself was first introduced in contemporary philosophy and psychology by Ehrenfels in his work *Über Gestaltqualitäten* (*On the Qualities of Form*, 1890). The central principle of gestalt psychological theory is that the mind forms a global whole with self-organizing tendencies. This principle maintains that the human mind considers objects in their entirety before, or in parallel with, perception of their individual parts; suggesting *the whole is greater than the sum of its parts*.

understanding of insight as the "interconnection based on properties of these things in themselves" (Koestler, 1964: 584). In the words of Koestler:

> "The pattern… is, *the perceiving into situation or Idea, L, in two self-consistent but habitually incompatible frames of reference, M_1 and M_2.* The event L, in which the two intersect, is made to vibrate simultaneously on two different wavelengths, as it were. While this unusual situation lasts, L is not merely linked to one associative context, but *bisociated* with two." (Koestler, 1964: 35)

Consequently, the creative leap or "an immediate perception of relations" can take place only if we are participating in at least two different frames of thinking, or matrices of discourse. The following quote, taken from Einstein's autobiographical notes (Schilpp, 1949:7) informs us how, in general, this "immediate perception of relations" takes place in the mind of the scientist, in agreement with Koestler's definition:

> "What exactly is thinking? When at the reception of sense impressions, a memory picture emerges, this is not yet thinking, and when such pictures form series, each member of which calls for another, this too is not yet thinking. When however, a certain picture turns up in many of such series then – precisely through such a return – it becomes an ordering element for such a series, in that it connects series, which in themselves are unconnected, such an element becomes an instrument, a concept." (Koestler, 1964: 7)

Cultural Values of Bisociation

Bisociation has the power, together with the construction of the schema of a new concept through "the immediate perception of relations", to transform a habit into originality in agreement with the Koestler's battle cry "<u>The act of creation</u> is the <u>act of liberation</u> – it's the defeat of habit by originality!" (Koestler, 1964: 96). Thus bisociation plays a dual role, that of a cognitive reorganizer and that of an effective liberator from a habit - it's planting double roots for creativity. The confirmation of the role as the affective liberation can be glimpsed from the research of Liljedahl, (Liljedahl, 2009: 213) "Aha experience has a helpful and strongly transformative effect on a student's beliefs and attitudes towards mathematics…". The Unity of Cognitive Reorganization with Affective Liberation is the characteristic quality of the Act of Creation – one of the new central cultural values brought forth by the new definition of creativity in mathematics.

The second new central cultural value brought forth by bisociation is the emphasis on the Democratization of Creativity. The quest for the democratization of creativity is the response to the seeming preoccupation of educational profession with the creativity of gifted children. There are two recently published excellent collections of papers, dealing with creativity in mathematics education (Sriraman and Lee, 2011; Leikin et al., 2009). Both collections join the issue of creativity with the education of gifted students, implicitly indicating that the interest in creativity of <u>all learners</u> of mathematics <u>is not the central focus of the field</u>. There can be several reasons for such a restrictive focus on creativity: it could be due to the efforts of globalization[3] so that "the winds are changing" (Sriraman and Lee, 2011: 2) or it could be that our understanding of the creative process is not sufficiently sharp to allow for the effective focus of research on the mathematical creativity by all students including, of course, the gifted. The proposed definition of creativity as the "spontaneous leap of insight," brings forward the 'Aha moment', easily observable within a general population, as the basis on which to investigate the creativity of all students. Both, the investigation into creativity and its facilitation for all rest on two following observations.

3 In approximate terms, globalization implies the freedom of capital, which is attracted by high rate of profit. Translated into didactic of mathematics education, capital is interested in new profitable discoveries, what leads to interest in giftedness as condition which promises higher than standard achievements and related profits.

1. Statement of **Hadamard** (Hadamard, 1945: 104): "Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention, one can say that there is only the difference of degree, the difference of a level, both works being of similar nature".
2. **Koestler** "minor subjective bisociation processes…are the vehicle of untutored learning" (Koestler, 1964: 658).

Since minor subjective bisociations are the standard vehicle of self-learning experienced by everyone, and **since** their nature is similar to that of the mature mathematical inventor, we can **therefore** view bisociation as the process that underlies creativity in mathematics for all – this suggests its usefulness in defining creativity in general.

The method of facilitating creativity in the classroom is suggested by the essential component of its definition "…connects previously unconnected frames of reference and makes us experience reality on several planes at ones" (Koestler, 1964:45). It suggests organization of learning environment along the interface of at least two intuitively (or better, habitually) unconnected frameworks such as geometrical line and real numbers, simultaneous discussion of several different representations of fractions in the context of operations (Prabhu et al., 2014) or along elementary algebra/ESL interface (Czarnocha, 2014a). Working along such bisociative frameworks increases the probability of "leaps of insight" both by students and teachers (examples below).

Moreover, since according to Koestler "minor subjective bisociation processes are the vehicle of untutored learning" (Koestler, 1964: 658) we need to create classroom conditions which approximate conditions of untutored learning which is best obtained by the discovery (or inquiry leading to discovery). Hence, the nature of bisociation provides a theoretical justification for the discovery method developed to its natural completion by the Texas Discovery method of R. L. Moore in US (Majavier, 1999).[4]

Below we provide three classroom examples of such pedagogy. The first one was constructed and implemented by Prabhu (Prabhu, 2014) in her classes of statistics and algebra. It used the notion of Koestler triptych and its aim was to consciously focus student attention on the interface between two related concepts to bring forth the "hidden analogy" between them.

Bisociative Facilitation of Creativity

Design of Triptych – Based Assignments

The Act of Creation (Koestler, 1964:45) defines bisociation, that is, "the creative leap [of insight], which connects previously unconnected frames of reference and makes us experience reality at several planes at once." Consequently, the creative leap of insight, or bisociation, can take place only if we are considering at least two different frames of reference or discourse.

How do we facilitate this process? Koestler offers a suggestion in the form of a triptych, which consists of "three panels…indicating three domains of creativity which shade into each other without sharp boundaries: Humour, Discovery and Art" (Koestler, 1964: 27).

4 Discovery method of teaching consists in creating learning environment which allows for student discovery of chosen mathematical concepts. Texan Discovery method generalizes that approach to full curriculum of graduate courses such as calculus, point set topology, euclidean and non-euclidean geometry e.t.c. A full course of Freshman calculus of the Texan discovery method might be a set of circa 150 axioms and theorems to prove, which leads the student to the discovery of all fundamental concepts of that subject. The level of student engagement parallel the engagement of participants in post graduate research seminars in European universities.

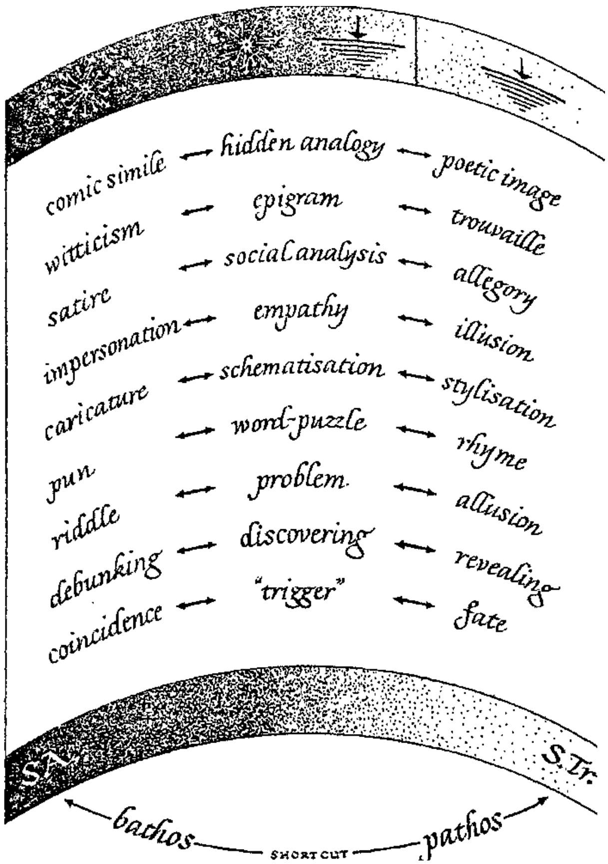

Figure 1. Koestler Triptych

Each such triptych stands for a pattern of creative activity, for instance:

Comic Comparison ⇔ Objective Analogy ⇔ Poetic Image

The first is intended to make us laugh, the second to make us understand, and the third to make us marvel. The creative process to be initiated in our developmental and introductory mathematics urgently needs to address the emotional climate of learners, and here is where the first panel of the triptych comes into play - humour. Having found humour and the bearings of the concept in question, the connections within it have to be explored further to "discover" the concept in detail, and, finally, to take the students' individual breakthroughs to a level where their discovery is sublimated[5] to Art.

Here's an example of the triptych assignment used by V. Prabhu (Prabhu, 2014) in her Introductory Statistics class:

Trailblazer ⇔ Outlier ⇔ Originality
⇔ Sampling ⇔
⇔ Probability ⇔
⇔ Confidence Interval ⇔
⇔ Law of Large Numbers ⇔
Lurker/Lurking Variable ⇔ Correlation ⇔ Causation

The triptych below is an example of student work:

Trailblazer ⇔ OUTLIER ⇔ Original
Random ⇔ SAMPLING ⇔ Gambling
Chance ⇔ PROBABILITY ⇔ Lottery
Lurking Variable ⇔ CORRELATION ⇔ Causation
Testing ⇔ CONFIDENCE INTERVALS ⇔ Results
Sample Mean ⇔ LAW OF LARGE NUMBERS ⇔ Probability

Triptych assignments facilitate student awareness of connections between relevant concepts and, thus, further support understanding. However, what maybe even more important is the accompanying discussions that help break the 'cannot do' habit and transform it into original creativity; below is the triptych completed by a student from a developmental algebra class:

Number ⇔ Ratio ⇔ Division
Part-Whole ⇔ Fraction ⇔ Decimal
Particularity ⇔ Abstraction ⇔ Generality
Number ⇔ Variable ⇔ Function
Multiplication ⇔ Exponent ⇔ Power

5 Sublime transformation of scientific discovery to art means its artistic refinement, which inspire admiration or awe. For example the conceptual art of the sixties and seventies was the sublimation of mathematical concepts of geometry, World Trade Centre towers built in the seventies represented, among others, sublimation of straight line.

The use of triptychs brings back the game and puzzle-like aspects inherent in mathematics into the mathematics classroom. What is the connection between the stated concepts? What other concepts could be connected to the given concepts? Given the largely computational nature of the elementary classes, and the students' habit of remembering pieces of formulas from previous exposures to the subject, a forum for making sense and exploring meaning is created to help connect prior knowledge with new synthesized reasoned exploration. The question 'how', answered by the computations, is augmented with the 'why' through the use of mathematical triptychs (Prabhu, 2014).

The second example reports the presence of the 'Aha moment' during the discussion of the process of solving a linear equation between two classmates (Poland, 3rd grade). Originally interpreted with the help of metonymy[6], it is an example of bisociation between perceptual and logical frames of thinking.

Perceptual-logical Resonance

The investigations into perceptual-logical resonance have been carried on by the group of teacher-researcher around Roberto Tortora and Maria Mellone from Federico Segundo University in Naples (Iannece et al., 2005). The example of the resonance has been provided by Polish educators, Wacek Zawadowski and Celina Kadej (Kadej, 1999) and it has been organized on the principles of Theory of Resonance developed by the author (Czarnocha, 2014b; Prabhu and Czarnocha, 2014).

An elephant – or what use can be made of metonymy?

Linear equations with one unknown can be solved by students in the elementary school in Poland. Those are simple equations and students often formulate them by themselves while solving word problems. Sometimes the problems lead to equations a bit more complex than the elementary additive equations of the type x + a = b.

I have had an opportunity to listen to the discussion of two enthusiastic students in third grade of the elementary school solving a standard word problem: The sum of two numbers is 76. One of the numbers is 12 more than the other. Find both numbers. It was a problem from a Semadeni's set of problems for the 3rd grade (Semadeni, 1987) and one had to solve it using equations and that's where the difficulty appeared:

Two pupils P & B's dialogue is given below. Pupil named P wrote the equation: x + (x+12) = 76. To solve it was a bit of a problem for him, but still he dealt with it. He drew an interval and then a following dialog had taken place:

P: That is that number: he extended this interval by almost the same length, and the another one like that.

And this is that number plus 12

B: and this all together is equal to 76…

P: No, this is an equation, d'you understand…

B: could not accept it…

B: Why did you draw this interval? You don't know yet what it's supposed to be?

P: That's not important.

B: Why 76?

P: 'cause that's what is in the problem

B: that x, that x add 12 and that's supposed to be 76..?

P: Look instead of x there is a little square in the book (P showed the little square in the book).

B: Aha, but here, here is written something else.

P: But it could be as here. And now I am inputting a number into this square.

6 Metonymy is a sudden change of meaning of the word or a symbol; here from "elephant" meaning itself to "elephant" meaning an unknown.

B: A number?! Why into the square?

P: No, it's into the window. Into this window I input the number which comes out here.

B: But here is a square (B insisted).

P: It's not a square but a window, and one inputs the numbers into that window.

B: How so?

P: Two windows are equal 64, one window is equal 32. Well, now, you subtract 12 from both sides, and you see that the two windows are equal to 64.

B: But are there numbers in the windows?

P: Two windows are 64, so one window is 32

B: Window!?

P: That's right, a window. Look here: **an elephant** and **an elephant** is equal **64**. Therefore what is **one elephant** equal to? **Two elephants** are equal **64**. So, **one elephant** is **equal** to what?

B: **An elephant**? Hmm, I see. **One elephant equals 32**. I understand now… so now the equation…

P: If **two elephants** are equal **60**, then **one elephant** is **equal** what?

B: An elephant?, ok, one elephant equals 30. I see it now…..Now equation…………..**aaaaaaa**

Thinking about that dialog one can have several questions: Why an elephant in P's thinking? Why window didn't work for B neither did line interval but an elephant worked? Where did the elephant come from?

There were two statutes on the bookshelf, a piggy and an elephant. P chose the elephant, ready to be taken as a symbol of some mental object. An elephant was used as an adequate symbol of a mental object, which often is called an x, but it doesn't have to.

It's a fascinating example of the 'Aha moment', where the bisociative framework is made up of the perceptual frame within which came the elephant, and of the logical frame of solving linear equations. More examples of the process can be found in Baker (2014b).

The domain of the function $\sqrt{X+3}$

The following example is from the remedial class of intermediate mathematics. The domain of the function $\sqrt{X+3}$ is at the center of the dialog.

Consider the square root domain question in the classroom of a teacher researcher, demonstrating the interaction between student and instructor, in which the latter is able to get the student engaged in the thinking process and hence to facilitate student creativity.

Note that it is the spontaneous responses of the student from which the teacher-researcher creates/determines the next set of questions, thus balancing two frames of reference, his/her own mathematical knowledge and the direction taken by the student. Similarly the student has her own train of thought and prompted by the teacher-researcher's questions, she must now balance two frames of reference to determine her next response. The analysis of the dialog is conducted with the help of bisociation theory and Piaget and Garcia Triad in Baker (2014a).

The problem starts with the function f(x) = $\sqrt{X+3}$

0. The teacher asked the students during the review: "Can all real values of be used for the domain of the function $\sqrt{X+3}$?"

1. Student: "No, negative X's cannot be used." (The student habitually confuses the general rule which states that for the function $\sqrt{X}$ only positive-valued can be used as the domain of definition, with the particular application of this rule to $\sqrt{X+3}$.)

2. Teacher: "How about $X=-5$?"

3. Student: "No good."

4. Teacher: "How about $X=-4$?"

5. Student: "No good either."

6. Teacher: "How about $X = -3$?"

7. Student, after a minute of thought: "It works here."

8. Teacher: "How about $X = -2$?"

9. Student: "It works here too."

A moment later

10. Student adds:" Those X's which are smaller than -3 can't be used here." (**Elimination of the habit through original creative generalization.**)

11. Teacher: "How about g(x) = $\sqrt{X-1}$?"

12. Student, after a minute of thought: "Smaller than 1 can't be used."

13. Teacher: "In that case, how about h(x) = $\sqrt{X-a}$?"

14. Student: "Smaller than "a" can't be used."(**Second creative generalization**)

Koestler defines a matrix as, "any pattern of behaviour governed by a code of fixed rules," (Koestler, 1964: 38) and, in statement (1) above, the limitations of the students' internal matrix, or problem representation, are demonstrated. The teacher, adjusting to the students' limited matrix provides two examples (lines (6) and (8)) that provide a *perturbation,* or a *catalyst*, for cognitive conflict and change. Recall that, as Von Glasersfeld (1989: 127) writes, "...perturbation is one of the conditions that set the stage for cognitive change".

In lines (6)-(9) the student reflects upon the results of the solution activity. Through the comparison of the results (records) they abstract a pattern, ⊠ "the learners' mental comparisons of the records allows for recognition of patterns" (Simon et al., 2004). Thus, in this example the synthesis of the student's matrix for substitution and evaluation of algebraic expressions with their limited matrix of what constitutes an appropriate domain for radical functions (bisociation) resulted in the cognitive growth demonstrated in line (10).

In lines (11) and (12), the perturbation brought about by the teacher's questions, leads the student to enter the second stage of the Piaget & Garcia's Triad. The student understood that the previously learned matrix or domain concept of radical functions, with proper modifications, extended to this example. They student was then able to reflect upon this pattern and abstract a general structural relationship in line (14), characteristic of the third stage of the Triad (Piaget & Garcia, 1983).

We propose the method of scaffolding presented above as the teaching-research inspired guided discovery method of creating a bridge between Koestler's insistence on the "un-tutored' nature of bisociation with Vygotsky emphasis on the socially structured nature of learning environment.

Bisociation and Simultaneity of Attention

Bisociation theory helps us to clarify certain ambiguities present in the professional discussion of mathematics creativity. The discussion here is grounded in the important Koestler's distinction between two kinds of thinking, progress in understanding reached through bisociation and exercise of understanding reached through the explanation of the particular case, through examining or using the coda formed by past experiences, both of which are defined below. Progress in understanding obtained through bisociation requires a new structure of attention which hasn't been discerned before in the field of mathematics education: that is the simultaneous attention towards two different frames of thinking. Identification of the simultaneity of attention as underlying bisociation brings us closer to the discussions of simultaneity in physics, both in Relativity theory and in the foundations of Quantum Mechanics. One may conjecture that similarly to the recognition of the fundamental character of simultaneity in physics, further research will demonstrate the fundamental nature of simultaneity of attention in the process of learning.

1. The standard division of creativity into absolute and relative[7] is misleading because it seems to suggest an essential difference between the two. Similarly, in each intellectual domain the tools and the language through which creativity is expressed vary, but the process of insight through bisociation is exactly the same. (See Section *Cultural Values of Bisociation*)
2. Situating the definition of creativity in the illumination stage of the Wallas definition itself provides a new perspective upon questions raised in recent discussions on the subject. In particular, Sriraman et al., (2011: 121) assertion can be qualified:

…when a person decides or thinks about reforming a network of concepts to improve it even for pedagogical reasons though new mathematics is not produced, the person is engaged in a creative mathematical activity".

Whether the process described above is or is not a creative mathematical activity can be decided on the basis of Koestler's distinction between progress of understanding - the acquisition of new insights, and the exercise of understanding - the explanation of particular events (p.619). Progress in understanding is achieved by the formulation of new codes through the modification and integration of existing codes by methods of empirical induction, abstraction and discrimination, bisociation.

The exercise or application of understanding to the explanation of particular events then becomes an act of subsuming the particular event under the codes formed by past experience. To say that "we have understood a phenomenon means that we have recognized one or more of its relevant relational features as particular instances of more general or familiar relations, which have been previously abstracted and encoded" (Koestler, 1964: 619).

7 Absolute creativity has a value at large, e.g. creativity of Sheakspire or Einstein; the relative creativity has a subjective value for an individual.

If for example, I decide to design the developmental course of arithmetic/algebra based on my knowledge of the relationship between arithmetic and algebra (generalization and particularization), which involves the redesign of the curriculum, that is of 'the network of concepts', I am engaged in the exercise of understanding of mathematics, distinctly different from creative progress of understanding in mathematics. It may however, depending on the initial knowledge of the teacher, be a creative activity in pedagogical meta-mathematics, that is understanding mathematics from the teaching point of view - the content of professional craft knowledge.

The bisociation theory, in which on the one hand, creativity is "an immediate perception of relation(s)", and on the other it is the affective catalyzer of the transformation of habit into originality, interacts well with MST methodology (Leikin, 2009). It predicts the absence of the difference between absolute and relative creativity observed by authors of the experiment. Moreover, the observed fall in the expression of originality reported by Leikin (2009), as well as the correlation between creativity and originality is natural in the context of the relationship between habit, creativity and originality – a point made explicit in the often quoted Koestler's assertion "Creativity is the Defeat of the Habit by Originality". The authors point correctly to the fluency and flexibility as the carriers of the habit which diminished the originality of student subjects "…when students become more fluent they have less chance to be original" (Leikin, 2009: 129-143). This apparently complementary relationship between fluency and creativity dictates an utmost care in conducting the research into creativity with the help of the definition which includes fluency, because it may result in undesired lowering of creativity. And that we don't want, especially in the 'underserved communities'. This observation brings in the old question to the fore: What is the optimal composition of fluency and creativity in the preparation of teachers of mathematics, as well as in classroom teaching?

1. The same distinction between the progress in understanding and the exercise of_understanding, helps us to clarify interpretation of an 'Aha moment' by recent presentation of Palatnik and Koichu (2014). The authors discuss the occurrence of the 'Aha moment' in the process of generalization of the numerical pattern obtained from calculating the maximal number of pieces that can be gotten from cutting a circle by straight lines as a function of the number of lines. There is an ambiguity in assigning the timing of the 'Aha moment', which we want to address from the bisociative point of view. We quote an extensive excerpt of their discussion:
2. Ron took the lead. In his words: "I was stuck in one to six. And I just thought…six divided by two gives three. I just thought there's three here, but I could not find the exact connection [to 22]. I do not know why, but I multiplied it by seven, and <u>voila</u> – I got the result."
3. One of these [Ron's] attempts began from computations 6:3=2 and 3x7=21. <u>Ron realized that in the second computation is not just a factor that turns 3 into 21, but also a number following 6 in the vertical pattern.</u> He noticed (not exactly in these words) the following regularity: when a number from the left column is divided by 2 and the result of division is multiplied by the number following the initial number, the result differs from the number in the right column by one. He observed this regularity when trying to convert 6 into 22, and almost immediately saw that the procedure works also for converting 4 into 11 and 5 into 16. He observed that even when division by 2 returns a fractional result (5:2=2.5), the entire procedure still works. <u>The aha-experience occurred at this moment.</u>
4. The bisociation theory suggests that the Aha moment took place a bit earlier, namely exactly at the moment when Ron observed the bisociative framework contained in the realization that 7 *in the second computation is not just a factor that turns 3 into 21, but also a number following 6 in the vertical pattern.* That was the spontaneous leap of insight, <u>the progress in understanding</u> of the problem by connecting two different frames – two different numerical patterns. The following computations quoted by the authors were already the result of the <u>exercise of understanding</u> reached in this Aha moment. We point out that partial responsibility for the absence of focus on that step as the fundamental one by the authors is borne by the Mason's theory of shifts of attention (Mason, 2008), which doesn't discern explicitly the simultaneity structure of attention needed for the bisociative leap of insight.

The introduction of the structure of simultaneity of attention raises some new research questions such as: What is the possible scope of simultaneous attention both in content and in time? One could also ask whether the scope of simultaneous attention is the same as the scope of the attention focused on a single object. If it is not the same, what is the dynamics through which attention focused on single objects transforms into the simultaneous attention upon both of them at once?

Measurement of Creativity

One of the chief reasons for the recent interest in Thorrace definition of creativity is its quantitative nature, which is therefore easily measurable. Fluency is measured by "the total number of interpretable, meaningful and relevant ideas generated in response to the stimulus"; while flexibility has been evaluated by "the number of non-repeating solutions" in students solution spaces. However, both, Koestler's theory of bisociation as well as the empirical results obtained by co-workers of Leikin (2009), Anat Levav-Waynberg and Raisa Guberman suggest that, the increase of flexibility and fluency, diminishes originality, and consequently creativity. Taking into account that, accordingly to Koestler Creativity is "the defeat of habit by originality", it follows that the development of habits to increase the number of solutions works against creativity itself. It follows that Torrace definition does not measure creativity, but the composition of creativity with the habit, conflating the measurement of creativity itself former.

Piaget-based APOS (Arnon et al, 2014: 66) theory of the conceptual mathematical development as explained by Baker (2014b) in which he had coordinated Koestler's bisociation theory with APOS theory, found out that interiorization and encapsulation, two basic reflective abstractions of APOS admit a bisociative framework. This implies that both of them can be realized through an 'Aha moment', a spontaneous leap of insight. Consequently the measurement of the creativity involved in the Aha moment could be the degree to which the particular reflective abstraction contributes to the development of the concept in question. We will obtain this way an answer to the question – How much did creativity of the 'Aha moment' contributed to learner's conceptual development?, and therefore to the progress of mathematical understanding.

Teaching-Research – the bisociative framework for teaching

The search for bisociative frameworks that is for two different, habitually incompatible (in Koestler's words) matrices of experience, using Koestler's description, leads are directly to the teaching-research methodology which underlie this paper. The original discovery of bisociation by Prabhu (2014) has taken place in the context of the teaching experiment *Problem Solving in Remedial Mathematics: Jumpstart to Reform* conducted in 2010/2011with the help of Teaching-Research/NYCity Model (Czarnocha, 2014c). Teaching-Research NYCity (TR/NYCity) Model is a methodology for classroom investigation of students' learning processes conducted simultaneously with teaching by the classroom instructor, the aim of which is a real-time improvement of learning in the very same classroom, and beyond. Since, as a domain, teaching is not habitually related to the domain of research, a teacher-researcher who is attempting to do both, acts within a bisociative framework, which is responsible for the large dose of creativity generated by that activity. Hence we see here bisociation as the single concept/process which can underlie both student's and teacher' creativity.

A full collection of examples and new creativity-based results obtained through the TR/NYCity Model is contained in the book *The Creative Enterprise of Mathematics Teaching-Research: Elements of Methodology and Practice – from Teachers to Teachers* to be published by Sense Publisher in 2015.

Conclusion

The presented discussion has proposed the new definition of creativity in mathematics based on the Koestler's theory of bisociation. We have analyzed the current, often used definitions of creativity utilized in mathematics education, and showed their limits as well possible negative effects on the development of creativity by students of mathematics. The discussion has led us through the characterization of relevant features of bisociation pointing out to the new cultural values of the unity between the cognitive and affective aspects of learning brought forth by the new definition. We have given limited examples of bisociation to the discussion of its classroom facilitation; many examples of bisociative thinking in humor, science and art are included in Koestler's *Act of Creation*. The analysis of the description of the particular 'Aha moment' by Palatnik and Koichu (Palatnik and Koichu 2014) suggested the importance to discern and to recognize simultaneity of structure of attention, which underlies simultaneity within the process of bisociation. Formulation of that structure opens many new research questions. The new qualitative measurement of creativity with the help of APOS theory has been proposed in the final pages of the essay but this still awaits the empirical verification.[8] Finally, the teaching research methodology with the help of which the presented results have been obtained was shortly presented as the creative bisociative framework for teaching. The concern for the coherence of the presentation motivated us to leave some important issues outside such as bisociation as the basis of new *computer creativity domain* (Berthold, 2012) and the relationship of mathematics with poetry as the bisociative framework.

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др Бронислав Чарноха

Хостос комјунити колеџ, Департман за математику, Универзитет града Њујорка, Сједињене Америчке Државе

О култури креативности у математичком образовању

Култура креативности у математичком образовању заснована је на дефиницијама креативности које се наводе у нашем истраживању и настојањима да се она спроведе у учионици. Међутим, не постоји јединствена дефиниција креативности у математичком образовању (Mann, 2006; Sriraman, 2005; Leikin, 2011, Kattou et al., 2011) која наставнику практичару ствара тачку ослонца у поучавању. Стога, култура креативности у математичком образовању не стоји на чврстим основама. Она сажима, између осталог, истраживање о креативности и истраживање о даровитости. Штавише, истраживања која је спровела група Леикинове (Leikin, 2009) открила су да нагласак на флуентности у размери креативног размишљања заправо умањује оригиналност, самим тим и креативност, што се слаже са Кестлеровим схватањем да је „креативност пораз навике од стране оригиналности“ (Koestler, 1964: 96). Из тих разлога су Прабуова и Чарноча научно полемисали на 38. међународној конференцији за психологију математичког образовања око прихватања бисоцијације Кестлеровог „Акта креације“, који је спонтани скок у унутрашњост као ауторитативна дефиниција креативности (Prabhu and Czarnocha, 2014). Овај рад представља теорију бисоцијације „аха“ момента, која се усредсређује на могућност елиминације инхибирајућих навика ума. Она истражује културне вредности које је донела нова дефиниција креативности, као што је демократизација, јединство креативности и мотивације у учењу уз подједнаку пажњу. Представљени су примери и методе примене у учионици. Разлика између бисоцијативног и асоцијативног учења наглашава увод у подједнаку пажњу као нову структуру пажње (Mason, 2008). Бисоцијација доноси са собом нове културне вредности: демократизацију истраживања и примену креативности, као и когнитивно-афективно јединство. Демократизација креативности заснива се на две поставке – Хадамарда и Кестлера. Хадамард изјављује (Hadamard, 1945: 104): „Између рада ученика који покушава да реши геометријски или алгебарски проблем и рада на проналаску, може се рећи да постоји разлика у степену, разлика у нивоу, а да су оба рада сличне природе“. Са друге стране, Кестлер (Koestler, 1964: 658) наводи: „Минорни субјективни бисоцијативни процеси [...] покретачи су учења које није вођено“. **Пошто су минорне субјективне бисоцијације стандардни покретачи самоучења кроз које свако пролази и пошто је њихова природа слична оној коју има зрео математички изумитељ, можемо да гледамо на бисоцијацију као на процес који потпомаже креативност у математици за све.** Бисоцијација је веома моћна идеја. Има моћ да заједно са конструкцијом схеме за нови појам „кроз непосредну перцепцију односа“ трансформише навику у оригиналност, што је повезано са Кестлеровим вапајем за борбу (Koestler, 1964: 96): „Чин креације је чин ослобођења – она је пораз навике од стране оригиналности!". Стога, бисоцијација игра двоструку улогу, ону која припада когнитивном реорганизатору и ону која припада ефикасном ослободиоцу од навике – то је усађивање дуплих корена за креативност. У истраживању Лилједала потврђена је улога коју може да наслути афективно ослобађање (Liljedahl, 2009: 213): „'Аха' искуство има помоћни и прилично трансформишући ефекат на веровања и схватања ученика према математици…" Јединство когнитивне реорганизације и афективног ослобађања је карактеристичан квалитет за чин креације – једна од нових централних културних вредности коју доноси нова дефиниција креативности у математици. Представљање бисоцијације као централног појма за разумевање креативности допушта нам да квалификујемо одређене погледе у професионалном математичком образовању. Нарочито могу да се квалификују изјаве Сримана (Sriraman et al., 2011: 121): „[…] Када особа одлучује или размишља о промени мреже појмова да би се она побољшала, па и из педагошких разлога, иако се нове математичке идеје нису формирале, особа је укључена у креативну математичку активност." Да ли процес, горе описан, јесте или није креативна математичка активност, може да се одреди на основу Кестлерове дистинкције између процеса разумевања – стицање нових сазнања – и вежбање разумевања – објашњење посебних случаја (Koestler, 1964). Напредовање у разумевању се постиже формулацијом нових кодова кроз модификацију и интеграцију постојећих кодова методима у емпиријској индукцији, апстракцији и способности разликовања, бисоцијацији. Вежба примене разумевања објашњења посебних догађаја постаје чин подсумирања посебних догађаја, а реализује се (у) кодовима које је формирало претходно искуство. Ако, на пример, одлучим да осмислим развојни програм аритметике/алгебре заснован на мом знању односа између аритметике и алгебре (генерализација и специјализација), ја сам укључен у вежбу разумевања математике, што се посебно разликује од креативног прогреса разумевања. Развојни програм укључује како редизајнирање курикулума, тако и редизајнирање „мреже појмова". Прогрес разумевања који се стиче кроз бисоцијацију захтева нову структуру пажње која није постојала у претходном пољу математичког образовања, а то је симултана пажња према два оквира размишљања. Идентификација симултаности пажње као потпора бисоцијацији ближе нас доводи до симултаних дискусија о физици заједно са њеним гранама и у теорији релативитета и у основама квантне механике.

Кључне речи: креативност, бисоцијација, "аха" моменат, симултаност пажње.

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Original Article

Snezana Lawrence[1], PhD
Bath Spa University, School of Education, United Kingdom

Mathematics Education in the Balkan Societies Up To the WWI

Abstract: *Whilst the world is indebted to the Greeks for their development of geometry and to Islamic mathematicians for their development of algebra, the history of violence and wars of the Balkan peninsula meant that neither heritages of these two great mathematical cultures survived into the 19th century. This paper is based on the research done for the history of mathematics in the Balkans and will be limited to the development of mathematical education in three Balkan societies: Greek, Ottoman, and Serbian, culminating in the early 20th century. It will try to explain how the three cultures of mathematics education were conceptualized, and how their development was influenced by the mathematical cultures of Western Europe. The systems of schools and universities, the first professors of mathematics at the universities in the three countries, mathematical syllabi, and some of the first textbooks in mathematics will be mentioned.*

Key words: *19th century mathematics education, Balkan mathematics, Greek mathematics, Ottoman mathematics, Serbian mathematics.*

Introduction

This paper focuses on the development of Balkan mathematics education, which will be covered through the history of three national schools of mathematics. In particular, it deals with identifying and describing the types of institutions in three societies of the Balkans in the latter part of the 18th and most of the 19th century. First, we will look at the mathematics of the ruling Ottomans, and examine how mathematical culture developed under the programme of modernization of the Ottoman state and their military apparatus. The second focal point will be the mathematics of two Orthodox populations then under the Ottoman Empire: Greeks and Serbs. Greeks were the predominant Orthodox *ethnie*, both in terms of their heritage and cultural influence, and the wide spread of the Greek diaspora, with merchant communities scattered throughout the Empire, gave them an enviable position in terms of their ability to import learning from foreign countries. Finally, we will look at the particular and relatively small national mathematical culture, that of Serbia, and examine a personal story that

1 s.lawrence2@bathspa.ac.uk

contains many ingredients considered typically Balkan, interwoven with a love of mathematical studies.

The Mathematics of the Ottomans

The Ottoman Empire (1299–1922) at the height of its power in the sixteenth and seventeenth centuries spread across three continents, from south eastern Europe, to north Africa and the Middle East, and included territories from Gibraltar to the Persian Gulf and from modern-day Austria to Sudan and Yemen. The relative religious tolerance meant that non-Islamic *ethnies* were allowed to profess their faith.[2] Until the end of the eighteenth century learning developed slowly in the Balkan principalities.

The Ottomans developed schools known as *madrasas*, (literally meaning 'place where learning is done) which were founded throughout the Muslim world from the ninth century. The primary aim of madrasas was to instruct in the science of jurisprudence. It is significant to note that certain madrasas included teaching on 'rational' sciences such as logic, ethics, Arabic language subjects, and arithmetic, apart from the religious and jurisprudence subjects. It is however, generally believed[3] that individual madrasas included mathematical sciences and astronomy if the madrasa professors studied such sciences and were therefore immersed themselves in such subjects.[4]

Students entered madrasas after their basic schooling in the *mektebs* (equivalent to primary schooling), and spent years rising through the ranks (twelve in total) corresponding to the ranks of their teachers. After completing studies in madrasas, a student may become a tutor, enter the teaching profession or, if all he had completed all grades, enter the *ulema* hierarchy becoming a learned man or a religious cleric. This type of education was based on tradition, and the handing on of existing knowledge, rather than the development of new concepts (Zilfi, 1983). The major challenge to this system, through wider political developments, came after the Ottomans lost the Battle of Vienna in 1683. The Ottomans subsequently fought a number of wars with the Russians, eventually establishing the Habsburg-Ottoman-Russian borders in southeast Europe. At the same time they lost large areas of the Empire, such as Egypt and Algeria, to Britain and France. The Ottoman government therefore became increasingly concerned by two major problems: the loss of power and influence on the one hand, and the perceived decline and corruption of the military apparatus on the other. There was hence seen to be an urgent need for modernizing of the army and the accompanying technology, which included changes in the learning of mathematics.

At its core, this modernizing process was not motivated by ideas of Western Enlightenment. Rather, the Empire sought to create an educational system geared towards meeting military needs by introducing Western engineering and scientific learning. The importation of western sciences therefore related to the art of war rather than to peace or the pursuit of knowledge itself. One of the major problems that eventually arose through this process was that of exclusion. The new educational programme that emerged, at least for the first half-century, and its close link with the military goals, excluded de facto the many ethnic groups whose culture was linked to Christianity rather than to Islam.

2 Roudometof (1998: 12) describes *ethnie* thus as a premodern concept of identity: 'An *ethnie* may have the following characteristics to differing degrees: a collective proper name, a myth of common ancestry, shared historical memories, some elements of common culture (e.g., language, religion), an association with a specific homeland, and a sense of solidarity'. See also Smith (1986: 40).

3 Under the reign of Kanuni Sultan Suleyman (1495-1566) this trend was most widely spread, but became less common after the end of the 16th century. See Sürmen, Kaya, Yayla, (2007).

4 See Lawrence (2008).

However, as the French and Turks had a long history of cooperation since the sixteenth century, when France received permission to trade in all Ottoman ports, the French and then other Western mathematics came into the Ottoman Empire eventually through this channel.

First, a Naval Engineering College was founded in Istanbul, at the Golden Horn – a fresh water estuary dividing old and new Istanbul – in 1773, under the guidance of Baron de Tott.[5] A Military Engineering College was established in 1795, with a mathematical syllabus almost identical to that of the Naval College, but with the additional subject of fortification. These two institutions were the first in the Empire to teach modern mathematics, focussing on the sciences and their application to military and civil engineering, and departing from the traditional Islamic teaching of the madrassas. The two colleges later merged and were, in effect, the origin of the Istanbul Technical University.

Among the first group of teachers at the Naval Engineering College was Gelenbevi Ismail (1730-1790), who is credited with introducing logarithms to the Empire. The second generation of teachers initiated a more organised programme of western mathematics by translating texts into Turkish. Huseyin Rifki, for example, who taught at the Military Engineering School became a prolific translator of western works. This was a part of the wide-ranging sentiment among the Ottoman scholars to increasingly look to the West rather than East, or to re-examine the Arabic mathematical heritage (Sürmen et al., 2007). Rifki's most important work, in collaboration with Selim Ağa, an English engineer who converted to Islam, was a translation of Bonnycastle's edition of Euclid's *Elements* (1789). Until this translation appeared in 1825, the teaching of Euclid came through the Arabic translation (c. 800) which was then modified by Bursali Kadizade-i Rumi (1338-1449) who was a head of Samarkand madrasa and published shortened and simplified versions of *Elements* under the title EsKalü't-te'sis (c. 1400).

Rifki's example tells of an overwhelming need to translate original works into national languages into from the western sources rather than from the existing translations of the original work. The practice indicates that the importation of western ideas of mathematics and of education corresponded with the production of new ways of learning even of the things that were already known. This phenomenon is an interesting one that perhaps requires further documentation and consideration, as it indicates that the western sources seemed to be deemed more suitable for the purpose of teaching at the newly established institutions of higher learning, rather than the original, native sources.

Perhaps this is because at the beginning of the nineteenth century Ottoman mathematics was strongly influenced by military preoccupations, and was mostly limited to the translation of French and English texts for use firstly, in the newly established military engineering colleges, and later for the use in teacher training colleges. The mathematics found in French and English sources was deemed modern for such uses. Likewise, the training into such mathematical culture became necessary. Kerim Erim was the first Ottoman mathematician to be granted a PhD in mathematics in 1919, at the Friedrich-Alexander University of Erlangen.[6]

As a consequence, because the local cultural tradition was abandoned in favour of the modern western mathematics (and it takes time for a community, as well as an individual, to become masterful in any discipline if learning is undertaken from the very beginning) very little of any original mathematics was done before the First World War. The second reason was also probably due to the mastery, or the lack thereof of the language skills to first learn, and then communicate results with the West-

5 According to Baron de Tott's memories, published upon his return to France in 1773. See De Tott (1785). De Tott was a French diplomat of Hungarian origin.

6 His thesis was entitled *Über die Trägheitsformen eines Modulsystems* as reported by İnönü (2006: 234-242).

ern mathematical communities. A rare case is that of a secondary school headteacher Mehmet Nadir, later becoming first professor of mathematics (1915) at the newly established University for Women (established in 1914 in Istanbul) and chair for number theory at the Istanbul University (1919), who obtained interesting results in number theory, particularly in relation to Diophantine equations (İnönü, 2006). Very little effort was also directed towards communication with Western mathematicians: Nadir did so, as did his contemporaries at the end of the 19th century – Tevfik Pasha and Salih Zeki both published in either foreign languages or in foreign journals – former wrote a book on algebra published in English under the title *Linear Algebra* (second edition 1893), for instruction at the teachers' colleges, and latter published an article on "Notation algébrique chez les Orientaux" in the *Journal Asiatique* in 1898.

Mathematics of Modern Greeks

The Greeks, after the collapse of the Roman Empire, found themselves under the Ottomans for centuries, but their diaspora spread as far as the Black Sea coast and the Venetian territories. Greek communities flourished in cities such as Vienna, Amsterdam, or Budapest on the one hand, and played an important role not only in commercial development, but in supporting the intellectual and cultural progress of Greeks and other Orthodox ethnies within the Ottoman Empire on the other. They did so by publishing books and newspapers in Greek, and enabling local Greek teachers to undertake university studies in the West.

One example of this practice was the education of Evgenios Voulgaris in the universities of Venice and Padova, supported by the brothers Lambros and Simon Maroutsis. Voulgaris focused on the re-establishment of ancient teaching in mathematics, believing classical Greek geometry to be the basis for any further progress (Roudometof, 1998). His view was that the uses of mathematics are valid in their interconnection with philosophy, rather than in experimental sciences. Thus the new ontological context of the Enlightenment escaped him, as it did some of his followers (Dialetis et al., 1997).

Voulgaris however also translated many philosophical and some mathematical works into Greek, among them, in 1805, Euclid's *Elements* from Tacquet's 1722 edition. *Elements* were first translated into Greek, albeit not of the 'modern Greek' variety, in 1533 by a German theologian and scholar Simon Grynäus (1493-1541). This edition included Proclus' *Commentary* on the first book of Euclid's *Elements*, given to Grynäus by the then president of Magdalen College Oxford, John Claymond.[7] The original manuscripts of ancient Greek origin were, by this time, not available.[8]

The areas around Thessalonika, the eastern Aegean, and the Ionian islands were the most significant in introducing new educational trends into Greek culture at the beginning of the nineteenth century, especially into mathematics. The intellectual prestige of these areas was related to sea trade which brought with it relative prosperity and the exchange of new ideas about learning. The centre of this new learning in Thessaly became Ampelakia, at the foot of Mount Olympus, where a school was founded in 1749. In the Aegean, the centres were Chios, Kydonies, and Smyrna. Academies were founded in Kydonies and Smyrna in 1800 and 1808, respectively (academies offering the equivalent of the undergraduate studies). The Ionian Islands were under British protection between 1814 and 1864, during which time the Ionian Academy,

7 This edition is now at the Brown University Library, US.

8 Kastanis (2006: 7) said of this: "It is well known that all Byzantine manuscripts of ancient Greek origin were pillaged, destroyed, or sold after the fall of Constantinople (an abomination beginning with the crusades from 1204-1261). Thus, the scientific works of the Greek civilization of antiquity, like Euclid's *Elements*, were missing both in the libraries of Neo-Hellenic communities and in those of the Orthodox monasteries, and it was extremely difficult for Greek scholars to access them."

established in 1824, introduced a Western model of mathematical education. Examples of teachers who taught mathematics in these institutions after training abroad are Veniamin of Lesvos, who studied at Pisa (supported by the Greek community at Livorno) and then taught at Kydonies from 1796, Dorotheos Proios who studied in Pisa and Paris and after 1800 taught at Chios (Kastanis and Kastanis, 2006), and Ioannis Carandinos, who studied at the École Polytechnique in Paris and later became Dean of the Ionian Academy.

The first translations into Greek of modern works on mathematics were done by Spyridon Asanis, a medical doctor who taught mathematics at Ampelakia in the 1790s. His translations drew on work by Nicolas-Luis de Lacaille (1712-1762) and Guido Grandi (1671-1742), and two of them were published: *Arithmetics and algebra*, in Venice in 1797, and *Conic sections*, in Vienna in 1803.[9] The success of these two books encouraged several further translations by others.[10]

Lacaille's work was then taken upon at the Greek Academy in Jassy by Iosipos Moisiodax (1730-1800). Lacaille was also very popular in both Italy and Austria during the second half of the eighteenth century, being introduced into their educational systems by the Jesuits (Kastanis and Kastanis, 2006). As majority of the Greek translations of western works were done through Venice it is conceivable that this is how Lacaille was introduced into Greece too.

Carandinos, who set up an undergraduate mathematics department at Corfu translated all the French books that he thought necessary and similar to those used in similar institutions in Western Europe. Between 1823 and 1830, he translated works by Bourdon, Biot, Lagrange, Poisson, Monge, Lacroix, and Legendre (Kastanis, 1998; Phili, 1998). Hence virtually all the mathematics studied at the Academy was pursued through the work of French mathematicians. Although French mathematics predominated in the Greek academies of the time, there were also other influences. Constantinos Koumas (1777-1836), for instance, studied at Vienna from 1804 to 1808 and completed his doctorate at Leipzig (Kastanis and Kastanis, 2006). His approach to mathematics is described as 'Austrian scholastic' (Kastanis and Kastanis, 2006) in as much as that his main focus in studying mathematics was made based on the work of Jean-Claude Fontaine, and he published in an eight volume work done by Fontaine and published in Vienna in 1800.[11]

The German influence could be traced between 1810 and 1820, and was brought into Greece by two mathematics teachers, Stefanous Dougas (1765-1829)[12] and Dimitrios Govdelas (1780-1831).[13] They introduced a German-inspired educational environment into the Patriarchic School of Constantinople and the Academy of Jassy. Bavarian officials also influenced the Greek educational system after the appointment of the Bavarian prince Otto Wittelsbach as King of Greece in 1832, to some extent. They established a system of secondary education divided into lower and upper Gymnasium, introducing a first syllabus in mathematics which gave much freedom to the teacher but prescribed

9 The original works were de Lacaille (1741) and Grandi (1744).

10 See Kastanis and Kastanis (2006: 518-520). Algebra was then extended by the study and publications of three scholars: Zisis Kavras (1765-1844) who studied in Jena and translated works from German; Dimitrios Govdelas (1780-1831) who studied in Pest and wrote a volume on Algebra relying on German sources; Stefanos Dougas (1765-1829), student of Halle, Jena and Göttingen, who published a four-volume work on arithmetic and algebra inspired by German tradition in Vienna in 1816.

11 Koumas (1807), Fontaine (1800).

12 See Dougas (1816). Dougas studied at Halle, Jena and Göttingen, publishing upon his return to the Balkans a four-volume arithmetic and algebra based on his learning of mathematics in these German cities.

13 Govdelas studied in Pest, where he wrote a volume on algebra. This work was published in Halle in 1806, under the title *Stoicheia Algebras* (Elements of Algebra). Upon his return to the Balkans he wrote a book on arithmetic and published it in Jassy in 1818. See Kastanis and Kastanis (2006: 519).

the general outline of study and the number of hours taught in schools. The emphasis was on classical studies, although mathematics was placed as a third most important subject after ancient Greek and Latin. Teaching of mathematics was heavily dominated by the teaching of geometry based on Euclid, and on Diesterweg's principles of teaching geometry based on heuristic or discovery learning.[14] The insistence on classicism and the fact that there was not enough mathematics teachers to populate the system, meant that there was a need for suitable textbooks, which were provided through the Bavarian connection. Georgios Gerakis for example, originally a teacher in a school in Athens, after his studies in Germany (enabled by the support from the state) published textbooks in Greek based on German textbooks: *Elementary Geometry and Trigonometry* (1842),[15] *Arithmetic and Algebra* (1855), and *Plane Geometry and Stereometry.*[16]

Serbian Mathematics

The history of Serbian mathematics is inextricably interwoven with the colourful lives of several of its most prominent exponents. Apart from tracing their stories, and as this special edition of the journal originates from within Serbia, it is quite interesting for us to see the origin of the Serbian modern mathematical culture.

Serbian mathematics education in the nineteenth century developed first under the rule of the Ottomans, and after 1833 under the Austro-Hungarian Empire. The first book on mathematics in Serbian was *Nova serbskaja aritmetika* (New Serbian arithmetic) (1767) by Vasilije Damjanović, but undergraduate education was established only in 1838, at the Lyceum in Kragujevac. The first mathematics professor there was Atanasije Nikolić, who had studied in Vienna and Pest, and his initial task was to write the first undergraduate textbooks in the Serbian language.

Belgrade University grew out of a succession of institutions, the most prominent being Matica Srpska, literally 'the Serbian Queenbee', founded in 1826 in Pest to promote Serbian culture and science. This institution grew into the Lyceum, and the Lyceum developed into the Superior School. The first trained mathematician to teach at the Lyceum, Dimitrije Nesić, had been educated at Vienna and Karlsruhe Polytechnic, and is credited with defining Serbian terminology for all mathematical concepts and processes known at the time.

At the end of the nineteenth century several Serbian mathematicians studied for doctorates at Western universities: Dimitrije Danić at Jena (1885), Bogdan Gavrilović at Budapest (1887), Djordje Petković at Vienna (1893), Petar Vukićević at Berlin (1894), and finally, the most famous Serbian mathematician, Mihailo Petrović, who completed his thesis in the same year (1894) in Paris. It is not known why Petrović chose Paris when all his contemporaries had studied in Germany or Austria, but he established important links with the French government during his studies and maintained them later. Thus, although most educational influences in the middle of the century were Austro-Hungarian or German, the most prominent of Serbian mathematicians, who set the future direction of the national mathematical school, introduced French mathematics and French mathematicians to his country.

Petrović, who was from a well-to-do family in Belgrade, completed a degree in natural sciences at the Superior, or sometimes called the Great School in Belgrade in 1889. He then went on to study at the

14 Friedrich Adolph Wilhelm Diesterweg (1790-1866) was a German education thinker whose most famous work, *Wegweiser zur Bildung für deutsche Lehrer* (*A Guide to Education for German Teachers*) (1835) set out the principles of teaching based on theory of development and improvement, heavily coloured by the ideology and philosophy of neo-Classicism. See Günther (1993).

15 The original textbook being Snell (1799).

16 Both original works were written by Carl Koppe (1803–1874). See Koppe (1836) and (1836a).

École Normale, originally a teacher training institution, rather than the École Polytechnique, which earlier in the century had been the preferred place of study for Greek students. The raised prestige of the École Normale at the end of the nineteenth century may have been a deciding factor, but it is not clear whether Petrović was aware of it. He was awarded his doctorate in 1894 for a thesis was entitled Sur les zéros et les infinis des intégrales des équations différentielles algébriques. The examining commission consisted of Hermite, Picard and Painlevé. Both Petrović and Painlevé later gained friends from the political elites of their respective countries. In 1906 Painlevé became a Deputy for the fifth arrondissement, the so-called Latin Quarter. He later became Prime Minister twice, in 1917 and 1925. Petrović on the other hand, became first tutor, and later good friend of the Crown Prince George Blackgeorge (Djordje Karadjordjević, 1887-1972). Petrović and Painlevé continued their friendship upon the return of Petrović to Belgrade. At Petrović›s insistence Painlevé›s work on mechanics[17] was translated by Ivan Arnovljević and published in 1828 in Belgrade as a textbook under the title *Mehanika* (Mechanics).

Petrović also made friends with Charles Hermite, who had already had another Serbian student, Mijalko Ćirić. Hermite taught Petrović higher algebra, and his son-in-law, Emil Picard, was another of Petrović›s examiners. Petrović and Picard became life-long friends, and Picard drew on work from Petrovitć›s thesis in his Traité d›Analyse (1908).[18]

Upon his return to Belgrade in 1894, Petrović was made a professor at the Superior School in Belgrade. At the beginning of 1905, the Superior School was replaced by the University of Belgrade and Petrović was appointed to the Chair in Mathematics, a position he held until his death in 1943.

Serbia changed relatively rapidly from having little or no mathematical culture at the beginning of the nineteenth century. There were some advantages to this relatively short history. At the International Conference on Mathematics Teaching (La Conférence International de l'Enseignement Mathématique) in Paris in April 1914, the Serbian representation reported that the introduction of infinitesimal calculus into schools was devoid of problems in their country, modernization did not pose a problem in a place where there was no tradition which could inhibit it:

> 'Chez les nations qui ont à peine dans leur développement, passé les premiers seuils de la civilisation, il n'y a pas de tradition et une idée en general et surtout une idée nouvelle, devient très facilement l'idéal meme d'une generation. Par consequent, dans ces circonstances la realisation de cet ideal n'est pas empêchée ou retardée par des questions de tradition'.[19]

Petrović's work, both in terms of acknowledgement in the international community and his efforts to establish a national school (virtually all mathematical doctorates in Serbia between the two World Wars were done under his supervision)[20] established far-reaching change. This had a long term

17 Painlevé (1922).

18 Trifunović (1994: 27).

19 L'Enseignement Mathématique: "With the nations which are, but at the threshold of civilization in their development, there is no tradition and an idea in general and especially a new idea, can become very easily an ideal of a new generation. As a consequence, in such circumstances the realization of this ideal is not prevented or delayed by the questions of tradition." (1914, 16, 332–333).

20 Petrović's doctoral students were Sima Marković (gained PhD 1904, became a famous Communist and as such disappeared and lost his life in Russia under Stalin), Mladen Berić (1912), Tadija Pejović (1923), Radivoj Kašanin (1924, who became professor at the University of Belgrade), Jovan Karamata (1926, who taught mathematics at the universities of Belgrade, Götingen, and Geneva), Miloš Radojčić (1928, professor at the University of Belgrade and the University of Khartoum), Dragoslav Mitrinović (1933, professor of mathematics and founder of mathematical institutes in a number of universities of former Yugoslavia), Danilo Mihnjević (1934), Konstantin Orlov (1934, professor of mathematics at the University of Belgrade), and Dragoljub Marković (1938); these mathematicians jointly produced further 361 doctoral students during their professional lives.

effect on Serbian, and later Yugoslavian, study of mathematics in the first half of the twentieth century. In this way the influence of the French school, was felt long after his main Serbian student became the founder of the national mathematical school.

Conclusion

The modern idea of a periphery, and in particular of societies such as those of the Balkans, lagging behind the West, which by contrast is seen as progressively onward-moving (Ahiska, 2003; Heper, 1980), implies a need to catch up with developments at 'the centre' by introducing new technologies, approach to studying sciences and mathematics, and the cultural innovations originating in the West. In the case of the Ottoman Empire, the sense of being on the periphery began to emerge at the end of the seventeenth century, which marked the beginning of the Empire's decline military prowess and influence. It was also, however, the beginning of the period during which the Ottomans began the process of Westernisation, including the adoption of Western mathematics, which entered Ottoman education mainly through the military engineering schools (Güvenç, 1998; Grant, 1999; Somel, 2001; Ekmeleddin, 2003; Gökdogan, 2005). The Ottoman effort to modernize the military, engineering, and mathematics, is an example of a periphery to which contemporary mathematics was brought and disseminated with a singular purpose in mind. In this case the periphery took what it considered useful from the West with a view to regaining military and political prestige, but filtered out other aspects of imported culture. However, and because the local cultural heritage was abandoned in order to adopt the modern, the catch-up process took a long time and the language barriers did their part in keeping the advances at a slow pace.

In the case of the Greek mathematics of the nineteenth century, the pursuit of mathematics was influenced by the centre to such an extent that the centre often set the agenda for reform in the periphery. In the case of modern Greek mathematics this is quite an extraordinary development to be seen, as virtually all modern mathematics of the 19th century still relied very much on Greek tradition of mathematical thinking. Nevertheless, the original Greek mathematics was lost, and the modern Greeks were re-learning mathematics from the West. This is seen in the mathematics exported by the French to the Ionian Islands, which at the time were a British protectorate. The mathematics developed at the Ionian Academy in turn impregnated all future developments in Greece after the wars of independence in 1821.

Finally, Serbian mathematics described here focuses on the small national mathematical culture. In this case, we mainly looked at one mathematician, who can probably described as the father of Serbian mathematical culture, Mihailo Petrović Alas. His deep interest in life led him to explore the North Seas as much as to search for authentic approach to doing mathematics. Alas counted among his friends the President of France, Crown Prince of Serbia, and the Gypsy musicians of Belgrade.[21] In such a setting mathematics and its narrative became embedded in the national culture, certain elements of which gave rise to an archetypal view of mathematical pursuit linked to a bohemian but also intellectually superior way of life.

With the First World War, the intellectual map as well as the political map of the Balkans changed dramatically. First, the centres of political and cultural influence changed drastically after the disintegration of the Austro-Hungarian Empire; second, whilst the choices mathematicians and mathematics educators made in the nineteenth century were often a matter of opportunities, beliefs into the routes into progressive new times, inheritance, or circumstances. The mathematicians of the new era became acutely aware of the seriousness of decisions they had to make in developing their national schools, an

21 See Lawrence (2008).

example of which is Petrović. Through this awareness they began to create their own intellectual landscape, and drew from the influences they considered most appropriate to their national circumstances.

With the Russian revolution of 1917, a new wave of changes swept through these lands, and with it the new mathematics brought by the refugees from the Czarist Russia. Many mathematicians that fled from the Russian revolution towards the west, stopped and settled in the Balkans. The first translation into Yugoslavian languages, for example, came through this route.[22] By the end of the Second World War the division of Europe into Western and Eastern Blocs meant further changes to the mathematical cultures of the Balkan societies, and the changes are yet to pan into the new stories for the new generations.

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др Снежана Лоренс

Педагошки факултет, Универзитет Бат Спа, Велика Британија

Математичко образовање на Балкану до Првог светског рата

Иако је цео свет захвалан Грцима за њихову геометрију и исламским математичарима за њихов рад на развоју алгебре, историја ратова и насиља на Балканском полуострву значила је да ниједна од ових двеју великих култура математике није преживела после 19. века. Овај рад заснива се на истраживању веза у историји математике на Балкану и ограничен је на разјашњење историје и развоја културе у три балканска друштва: грчком, отоманском и српском, а завршава се догађајима у раном 20. веку. Овим радом покушаће да се опише и покаже како су ове три културе математичког образовања концепутализоване и како је њихов развој био под утицајем математичких култура западне Европе. Описаћемо системе школа и универзитета, прве професоре математике на универзитетима у ова три друштва, као и њихове програме математике и неке од првих уџбеника математике.

Прва математичка култура коју описујемо у раду јесте отоманска, и то са позиције развоја њеног друштва и државе, те војног уређења и, наравно, математике, којом се бавимо у поменутом контексту. Отоманска царевина (1299–1922), на врху власти у 16. и 17. веку, ширила се на три континента, од југоисточне Европе до северне Африке и Блиског истока и обухватала је територије од Гибралтара до Персијског залива и од модерне Аустрије до Судана и Јемена. Отомани су развили систем школа – медресе, које су осниване од 9. века широм муслиманског света. У медресама су се, осим проучавања религиозних научних дисциплина, проучавале и дисциплине посвећене рационалним наукама, као што су арапски језик, логика, аритметика и етика. Рад прати развој отоманске математике у Царству, од медреса до првих универзитета, показујући утицај који су имали Французи и Енглези у успостављању школа у Царству, као и уџбенике који су се преводили са француског и енглеског језика и били коришћени у отоманским институцијама знања и учења.

Грчко математичко образовање, мада историјски вероватно има највећи утицај на развој математичког образовања у европском и западном свету, није имало континуитет на грчком подручју, које би повезало старогрчко и модерно грчко математичко образовање и културу. Грци су се, после колапса Римске царевине, нашли под отоманском владавином, која је трајала столећима. У раду пратимо како су Грци успели поново да успоставе своју интелектуалну и математичку културу кроз специфичност њиховог статуса под Отоманима. Наиме, Грци су били познати као највећа ортодоксна етничка група у Отоманском царству, и као такви имали су посебне привилегије и приступ владајућим Отоманима. Неколико примера који се могу пратити кроз историју модерне грчке математике показују утицаје под којима су се нашли на прагу свог ослобођења од отоманске владавине.

Посебна снага грчке културе у овом периоду била је њихова дијаспора. На пример, Вулгарис (Evgenios Voulgaris) завршио је универзитет у Венецији и Падови, што су му омогућила браћа из дијаспоре, Ламброс и Симон Марутсис (Lambros and Simon Maroutsis). Вулгарис се усредсредио да поврати својој домовини нешто од старе грчке математичке културе верујући да је грчка геометрија основа за било који будући напредак у математичком образовању.

Рад даље прати развој грчког математичког образовања, показујући нам да су француски, енглески и немачки утицаји били преовлађујујући у успостављању модерне грчке математичке културе и образовања.

Српска математика, мада релативно млада, од посебног је интереса за рад, не само зато што се наш рад налази у публикацији која потиче из Србије него и због специфичности релативно мале културе која је произвела важну и утицајну математику и математичку културу и произвела веома угледне математичаре у релативно кратком времену. Српска математика развијала се под утицајем Отомана, а после 1833. године под утицајем Аустроугарске монархије. Прва књига о математици на српском језику штампана је тек 1737. године, а студије математике на вишем нивоу настају тек 1838. године (Лицеј).

И поред тако касног почетка, на крају 19. века Србија је већ имала неколико добрих математичара на докторским студијама у Паризу, Бечу, Берлину и Будимпешти. Најпознатији од њих био је Михаило Петровић, звани Алас, који је у Паризу направио неколико важних контаката и веза са математичарима и политичарима, што је омогућило српској математичкој култури приступ важним скуповима, од којих је један била Интернационална конференција математичког образовања одржана у Паризу априла 1914. године, када је српска делегација дала извештај у коме је саопштила да је краткорочна историја некад погодна за напредак математике:

„Код оних нација које тек почињу свој напредак, без основа традиције, генерално идеје, а специјално нове идеје, могу постати важан идеал за нове генерације...“ (L›Enseignement Mathématique, 16 (1914), 332–333).

Историја српског математичког образовања и културе завршница је овог поглавља.

Кључне речи: математичко образовање у 19. веку, математика на Балкану, грчка математика, отоманска математика, српска математика.

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Mailizar Mailizar, Lecturer

Syiah Kuala University, Indonesia, PhD student University of Southampton, Southampton Education School, Southampton, UK

Manahel Alafaleq, PhD student

University of Southampton, Southampton Education School, Southampton, UK

Lianghuo Fan[1], PhD

University of Southampton, Southampton Education School, Southampton, UK

A historical overview of mathematics curriculum reform and development in modern Indonesia

Abstract: *Indonesia has the fourth largest education system in the world in terms of student population; yet due to a variety of reasons, internationally there is little literature available about Indonesian education, particularly in its historical change and development. This paper focuses on Indonesian national school mathematics curriculum, and provides a historical overview and documentation of the reform and evolution of the mathematics curriculum in modern Indonesia. Both external and internal factors in relation to Indonesian education that have influenced the mathematics curriculum reform and development in this period of time are examined and their implications to general mathematics curriculum reform and development are discussed in the paper.*

Key Words: *history of mathematics education, Indonesia mathematics education, mathematic curriculum reform and development.*

Introduction

Located in Southeast Asia, Indonesia has the fourth largest education system in the world in terms of student population. However, large-scale International comparative studies such as the Trends in Mathematics and Science Studies (TIMSS) and the Programme for International Student Assessment (PISA) have consistently shown that the Indonesian educational system does not work well in terms of students acquiring a good quality of education at the primary and secondary levels. For example, Indonesian 15-year-old students were placed 57th out of the 65 participating countries/territories in PISA 2009 in their average mathematics scores[2]. In PISA 2012, they were ranked 64th out of the 65 participating

1 L.Fan@southampton.ac.uk

2 See http://www.oecd.org/pisa/pisaproducts/46619703.pdf

countries/territories.[3] These indicators have been a driving force for the Indonesian government to undertake the latest national curriculum reform (Suryadarma & Jones, 2013).

To improve the quality of students' learning in any education system, it is essential to look at its curriculum, as curriculum is a prime part of that system and plays a vital role in determining why, what, and how students learn and are taught in schools. According to Levin (2008), curriculum is defined as an official statement of what students are expected to know and be able to do. Curriculum is particularly important in countries like Indonesia, which adopts a centralized education system.

In the history of modern Indonesia's education, the national curriculum has undergone many changes in the years 1947, 1952, 1964, 1968, 1975, 1984, 1994, 1999, 2004, 2006 and the latest is 2013. As to Indonesia's national school mathematics curriculum, Soedjadi (1992, as cited in Suryanto et al., 2010) once classified this long reform into the following eras:

1. before 1975
2. Era of modern mathematics
3. Back to 'tradition mathematics'
4. Integrated Era

However, literatures about Indonesian mathematics education are overall very limited particularly regarding the history of mathematics curriculum. This is so even since the 1970s, the policies of education reform in Indonesia have proceeded in the context of human resources expansion for the purposes of national development (Yeom et al., 2002), and moreover, there is a growing awareness among scholars in Indonesia of the need to improve mathematics teaching in schools (Sembiring et al.,2008).

In this paper, we look back at the history of mathematics curriculum reform and development in modern Indonesia, mainly through the national curriculum materials, policy documents and available literature. By doing so, we intend to provide a historical overview and documentation of the reform and evolution of mathematics curriculum in Indonesia, examine external and internal factors in relation to the curriculum reform and development in the country, and discuss their implications for further curriculum reform and development. We therefore start with a brief introduction about Indonesian mathematics curriculum before 1975, which we termed pre-modern mathematics curriculum.

Pre-Modern Mathematics Curriculum (before 1975)

Since Indonesia got its independence in 1945, mathematics as a school subject has been a compulsory course throughout the whole school education, that is, from primary school (Grades 1-6), to junior high school (Grades 7-9) and senior high school (Grades 10-12). However, before 1975, the teaching of mathematics was mostly influenced by Western mathematics education theories, and in particular, Skinner's behaviourism of learning (Ruseffendi, 1988). As Zulkardi (2002) noted, the lessons were delivered through mechanistic pedagogy. Students were trained to memorize mathematical concepts without understanding them (Ruseffendi, 1979). In learning geometry, for instance, Ruseffendi revealed that it was focused on developing calculation skills, and the students learned how to calculate area and volume of a geometric object without understanding the meaning of area and volume (Ruseffendi, 1979).

It should be noted that Indonesia's national mathematic curriculum before 1975 was implemented based on the separate mathematics strands such as algebra, geometry and trigonometry (Zulkardi, 2002). Regarding the contents of the curriculum, arithmetic was taught in the primary schools, algebra and plane geometry were taught in the junior high schools (Grades 7-9), while in the senior

3 See http://www.oecd.org/pisa/aboutpisa/pisa-2012-participants.htm

high schools students learned more advanced algebra, three-dimensional geometry, and analytic geometry. The main criticism of this curriculum was that it did not pay adequate attention to the relationship between different areas and topics of mathematics (Ruseffendi, 1979).

Modern Mathematics Curriculum (1975)

In 1973, the Indonesian government translated “Entebbe Mathematics Series”, which was developed in mid 1960s mainly by US and UK mathematicians and mathematics educators, and aimed mainly for the African countries (Williams, 1971). The translated series were then used as main mathematics textbooks in Indonesia. This translation project was the beginning of the implementation of modern mathematics in Indonesia mathematics education.

In 1975 the Indonesian government officially implemented a new curriculum which was deeply influenced by modern mathematics movement or “new math” (Kilpatrick, 2012; Sembiring et al., 2008). According to the Ministry of Education and Culture, or Depdikbud in Indonesian, the mathematics curriculum in this period was characterized by the following criteria (Depdikbud, 1976):

1. New topics were introduced;
2. More focus was placed on developing understanding rather than memorization and calculation skills;
3. Attention was paid to continuity among the topics in primary and high schools;
4. Heterogeneous or different students’ needs were accommodated;
5. Student-centred learning was emphasized.

The new topics included in the curriculum were *Set, Statistics, Probability, Relation and Function, and Non-Metric Geometry* (Depdikbud, 1976). Moreover, *Plane Geometry and Three-Dimensional Geometry* which were taught at different levels in the previous curriculum were taught at the same level, at year 11, in this curriculum.

With regard to teaching approaches, deductive approaches were used not only in geometry but also in algebra in high school. However, inductive approaches were still used for primary school students (Suherman & Winataputra, 1999). Moreover, according to Ruseffendi (1988), this period was strongly influenced by behavioural psychology that emphasizes the stimulus to response and training (drill). In addition, Piaget and Bruner’s theories also played an important role in shaping teaching approaches advocated in curriculum and classroom practices in this period (Ruseffendi, 1988).

Like in many other countries, it was also admitted that unfortunately in Indonesia also, the modern mathematics, which had been introduced into the curriculum since the beginning of 1975, resulted in a problematic situation in schools (Sembiring et al., 2008; Cockcroft, 1982). By 1983, this modern-mathematics-based curriculum was considered no longer suitable in order to meet the community’s needs and the demands of science and technology. New calls for a new mathematics curriculum ensued.

Technology-Integrated Curriculum (1984)

The Indonesian government decided to develop and implement a new curriculum starting from 1984. There were actually no significant changes in terms of the total coverage of mathematics topics in the new curriculum, as compared to the previous one (Depdikbud, 1987). However, three new features make this new mathematics curriculum particularly noteworthy.

Firstly, this curriculum signalled the first attempt and policy directive to integrate modern technologies into the mathematics teaching and learning in Indonesian classrooms. Most specifically, calculators were introduced into the teaching of mathematics. It is in this sense we call this curriculum “technology-integrated curriculum", though this was only a starting point in this direction. According to Ruseffendi (1988), this was one of important efforts in strengthening mathematics education in Indonesia.

Secondly, there was an important change in the sequence and structure of the mathematics contents introduced in the curriculum. For examples, some topics such as algorithms, trigonometry, and transformation were moved from the senior high school level to the junior high school level (Depdikbud, 1987).

Thirdly, a "spiral" approach as a pedagogy was adopted in the new curriculum. Table 1 below shows an example of how a concept of geometry (area) was packed in the curriculum (Depdikbud, 1987).

Table 1: An Example of Spiral Approach in Teaching Areas of Geometric Shapes

Grade Level	Topics
Grade 3	The students were introduced the ratios of area of a square and rectangle, and then they learned the area of a square and rectangle through counting square plot.
Grade 5	The students recalled what they already learned at Grade 3. Thus, they learned the area of a square and rectangle by multiplying the square plots on rows and columns; from this activity they learned the formulas of square and rectangle.
Grade 5	The students learned the area of a triangle.
Grade 6	The students learned the area of a parallelogram, then they were introduced the area of a circle.
Grade 7	The student recalled the concepts of the area of a square and a rectangle, and then they learned the area of a cube and block.
Grade 8	The students learned the area of a rhombus, trapezium and kite.
Grade 8	The student learned the area of a circle its application.

The "spiral" approach was reflected in the width and depth of learning materials, so that the higher the school levels, the more width and depth of the materials and lessons were provided on same topics.

Regarding teaching approaches, the Ministry of Education and Culture (or Depdikbud) recommended that the *Student Active Learning (Cara Belajar Siswa Aktif or CBSA* in Bahasa Indonesia) approach be adopted for learning and teaching in all schools (Depdikbud, 1987). CBSA is a teaching approach that provides the opportunity for students to be actively engaged in the learning process and with the hope that students get the maximum learning experience, in cognitive, affective, and psychomotor aspects (Pardjono, 2000). Internationally, this curriculum was mostly influenced by developmental psychology of Piaget (Flavell, 1967).

However, as Fauzan (2002) noted, the implementation of this new curriculum had also made clear a number of problems and in particular, the following:

1. An overload of subjects at the primary school level, which had resulted in the fact that the students often did not have sufficient time to master any given subject.
2. A lack of continuous assessment of the students' progress.
3. An unsatisfactory implementation of the active learning principles.

Therefore, all these problems had stirred up strong criticism from the parents and society (Depdikbud, 1997), a reason for the government to develop another new mathematics curriculum.

Back-to-Basic Curriculum (1994)

In 1994, the curriculum reform in Indonesia was signified by the change of curriculum content and teaching approaches, especially at the primary school level. In fact, as Armanto (2002) noted, the reformed curriculum in 1994 had made significant changes in many aspects compared with the previous curriculum launched in 1984.

Government reports (Depdikbud, 1994) indicated that the main aims of teaching mathematics in the 1994 curriculum were:

1. Students are able to effectively and efficiently deal with the dynamic world based on logical reasoning, rational and critical thinking.
2. Students are able to use mathematics and mathematical reasoning in studying other subjects.
3. Students have critical attitude, perseverance, and appreciation of mathematics.
4. Students understand mathematics deductively.

From the aims of teaching mathematics mentioned above, we can see that since 1994 the Indonesian mathematic curriculum already paid much attention to critical aspects of mathematics education such as developing students' reasoning and skills to deal with real life problems, which was not clearly stated in the previous curricula. These goals are similar to those stated by the National Council of Teachers of Mathematics (NCTM, 2000) that mathematic curriculum should prepare students for solving problem in a variety of school, home and work settings.

In order to achieve the main goals of teaching mathematics, specific instructional objectives were provided by the government for the teachers in the curriculum. The following is an example of specific instructional objectives as mentioned on GBPP, which is an Indonesian abbreviation of Curriculum Implementation Guide, published in 1994:

Table 2: An Example of Specific Instructional Objectives

General instructional objective	Specific Instructional Objectives
Students are able to measure the size of angles and areas, and to understand measurement units	1) Students are able to determine the area of squares and rectangles by counting the number of square units and/or by counting the number of square units in one row then multiplying it by the number of rows. 2) Students are able to recognise the formulas for era of squares and rectangles. 3) Students are able to recognise standard measurement units for area.

In practice, it is not always possible to precisely specify the instructional objectives for some of the main aims of mathematic teaching in a given topic, therefore the main aims could become blurred (Fauzan, 2002). For instance, in teaching geometry the specific learning objectives in the 1994 curriculum were focused on remembering definition of two and three dimensional geometric objects such as square, cubes, prisms, and memorizing the characteristics of these objects, but did not refer to more broad aims of learning geometry such development of logical reasoning ability (Suydam, 1983) or interpretation of space and the environment (Moeharty, 1993). It appears that the specific leaning objectives were not well aligned with the main aims of the teaching and learning of mathematics as mentioned earlier.

In terms of the mathematics contents, fundamental changes were observed for the new school mathematics curriculum. The emphasis was placed on students' mastery of fundamental principles of mathematics, particularly at the primary school level, in which the "traditional" mathematics with a focus on calculation skills again received more attention in this curriculum (Depdikbud, 1994), and some "modern" topics, for instance, the *Set Theory*, were no longer a focus in the curriculum (Armanto, 2002). It is in this sense we call this reformed curriculum "back-to-basic curriculum". However, the idea of going back to basic emphasized in the curriculum seems contradicting or incoherent with one of the main aims of the curriculum, that is, students were expected to be able to use mathematics and mathematical reasoning in their daily life. Moreover, at the senior high school level, the introduction to graph theory was included in the curriculum while integration was not. It would be interesting to see the reason behind these changes. Unfortunately we were not able to locate any literature regarding this issue, nor could we reach the curriculum developers to gather information and make clarification due to the scope of this study, a limitation warranting further effort in future study.

Content-Reduced Curriculum (1999)

As it had too heavy content for teachers and students to get through, the 1994 curriculum was later considered overloaded (Supriyoko, 1999). Moreover, as Supriyoko also pointed out, the 1994 curriculum was not flexible so the teachers were unable to find adequate room for developing students' creativity in teaching and learning activities. In addition, Fauzan (2002) noted that teachers complained about having too many topics, too limited time to teach them, and the students complained about having too many exercises and too much homework to complete in a school year. Therefore, the government decided to make some adjustments in the national mathematics curriculum.

The new mathematics curriculum was released by the Indonesian government in 1999. It is largely a simplification of the 1994 curriculum. One of the most important features of this new curriculum was reducing so-called irrelevant or unessential topics such as *sets* and *introduction to graph theory* (Fauzan, 2002). Unfortunately, we could not find any literature concerning why these topics in particular were regarded irrelevant by the government reformers at that time.

In addition, for this curriculum, the government only required all students to master core content. For those who were more interested in mathematics or *mathematical gifted* students the new curriculum offered advanced mathematical contents. This advanced content was managed and adjusted by teachers based on students' needs. Some of the advanced topics were, for example, *Measures of Skewness and Kurtosis*, *Inverse Function and Composition, the derivative and integral of the Exponential Function* (Depdikbud, 1999). The new curriculum advised that the content of mathematics taught and the levels of difficulty must be continuously reviewed and updated when necessary in order to meet students' needs.

It should be noted that content reduction in mathematics curriculum was also reported in many other countries especially in Asian countries including China, Japan, and Singapore around the same time (Bjork & Tsuneyoshi, 2005; Wu & Zhang, 2006). For example, in Singapore, the government announced in 1998 that there was a 10%-30% content deduction in most school subjects including mathematics with the purpose of providing room for teachers to implement the new initiatives in schools, such as the development of thinking skills, integrating the use of Information Technology, and the delivery of the National Education (Singapore Ministry of Education, 1998) in school education. In that sense, the reform in Indonesian mathematics curriculum was consistent with many other countries.

Concluding Remarks

Since the 1970s a number of studies (Haji, 1999; Jailani, 1990) have shown the weaknesses of mathematics teaching in Indonesia. Indonesian students find it difficult to comprehend mathematical concepts, and the teaching approaches commonly used in Indonesian classrooms make mathematics more difficult to learn and to understand. Moreover, the results of the national examinations showed that mathematics was continuously the lowest-scoring subject (Depdikbud, 1997).

From our discussion above, we can see that Indonesian mathematic curriculum reforms were to a large extent influenced by and consistent with the trends in other countries. For instance, when "modern mathematics" became the dominant movement around the world, the Indonesian government implemented a new curriculum framed by this new trend. Modern mathematics was rated highly and expected to provide Indonesian students with a good opportunity to learn mathematics more effectively (Sembiring et al., 2008). Unfortunately, and in practice, many teachers reported many problems with this approach as modern mathematics was too difficult for their students to learn (Somerset, 1997).

As researchers have noted, even though the curriculum reforms not only focused on mathematics contents but also on teaching approaches, the teaching and learning of mathematics in Indonesian schools remained mechanistic, with teachers tending to dictate formulas and procedures to their students (Armanto, 2002; Fauzan, 2002). Hence, it seems that the curriculum reforms over the last five or so decades failed to bring significant impact on classroom teaching and students' achievement in learning mathematics, as mentioned at the beginning of the paper. To us, this indicates the challenge and complexity of curriculum reform and development, and as a developing country with ambition to improve the teaching and learning of mathematics in schools, the case of Indonesia presents a meaningful lesson and example for learning and study that goes beyond its own geographical boundary.

Finally, regarding the Indonesian mathematics curriculum reforms, we think the following two issues are noteworthy as a conclusion to our paper.

First, we did not find that in the period of reforms examined in our study there was a mathematics curriculum framework at the national level that could guide the country's mathematics education community in reforming or changing the mathematics curriculum. Hence, with respect to the contents of mathematics in the curriculum, it seems that the change was to some extent not well planned and articulated during the period of the reforms. We also looked at other countries, and found that, for instance, Singapore has a well-established and articulated mathematics curriculum framework that was published in 1990, which has been used as a basic guidance in framing curriculum reforms in the country ever since.

Second, another important issue in reforming curriculum is about conducting a curriculum needs assessment. As Oliva (1991) noted, a curriculum needs assessment is a process that identifies programmatic needs that must be addressed by curriculum planners. However, concerning this issue, and as Sudiarta (2003) revealed, the Indonesian government had very weak needs assessments in reforming the curriculum. More generally, how governments can conduct a curriculum needs assessment in certain educational, economic and social context is an issue that merits further attention from mathematics curriculum reformers and developers in Indonesia and other countries especially developing ones like Indonesia with a centralized education system.

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Маилизар Маилизар
предавач, Универзитет Сија Куала, Индонезија,
докторанд, Педагошки факултет, Универзитет у Саутемптону, Велика Британија
Манахел Алафелек
докторанд, Педагошки факултет, Универзитет у Саутемптону, Велика Британија
др Лиангуо Фан
Педагошки факултет, Универзитет у Саутемптону, Велика Британија

Историјски осврт на математичку курикуларну реформу и развој у модерној Индонезији

Индонезија је четврта земља у свету из области образовног система и популације ученика, мада у свету постоји врло мало литературе о индонежанском образовању, нарочито оне која се тиче историјских промена и развоја од њене независности 1945. године. У овом раду се осврћемо на историју математичке курикуларне реформе и развоја у модерној Индонезији, пре свега кроз националне курикуларне материјале, документа у вези са националном политиком и доступном литературом, и даље – до историјског осврта и документације реформе и еволуције математичког курикулума у индонежанским школама. Заснован на нашем осврту и анализи, ова модерна историја индонежанског математичког курикулума може да се подели у пет фаза: 1) Предмодерни математички курикулум (пре 1975), који је превасходно био базиран на засебним математичким стандардима, као што је алгебра, геометрија и тригонометрија. У овом курикулуму није обраћано довољно пажње на односе између различитих математичких тема; 2) Модерни математички курикулум (1975), на који је много утицала модерна математика или „нова математика" наглашавајући структуралистички приступ. Као и у многим другим земљама, и у Индонезији је прихваћено да модерна математика, која је заснована 1975. године, доводи до проблематичних ситуација у школама; 3) Технолошки интегрисан курикулум (1984), који заправо нема битних промена у смислу опште покривености математичких тема у поређењу са претходним курикулумом. Мада су нове карактеристике овог курикулума следеће: прво, увођење калкулатора у курикулум је сигнал првог покушаја интеграције модерне технологије у математичко поучавање и учење у индонежанским школама. У овом случају, то називамо „технолошки интегрисаним курикулумом". Друго, постоји знатна разлика у следу и структури математичког садржаја у курикулуму. Треће, спирални приступ је педагошки приступ који је био усвојен у новом курукулуму; 4) Курикулум који се заснива на повратку на основе (1994). Курикуларна реформа у Индонезији 1994. године означена је променом у курикуларном садржају и наставним принципима, нарочито на основношколском нивоу. Од циљева учења математике, у овом курикулуму је већ обраћано много пажње на критичке аспекте математичког образовања, као што је развијање вештина резоновања и оних који се тичу стварних животних проблема, у поређењу са претходним курикулумом. У математичком садржају основне промене су начињене и нагласак је на усавршавању елементарне математике, нарочито на основношколском нивоу, на коме се више пажње посветило „традиционалној математици", са нагласком на вештине рачунања, а неке „модерне" теме, као што је, на пример, теорија скупова, нису више биле у фокусу курикулума. Зато га

називамо „курикулумом који се заснива на повратку на основе“; 5) Курикулум редукованог садржаја (1999) jесте ревизија претходног курикулума и настао је, пре свега, смањивањем броја математичких тема, јер је курикулум из 1994. године сматран претрпаним и недовољно флексибилним и наставници нису могли да пронађу довољно простора за развијање ученичке креативности у активностима учења и поучавања. То је поједностављење курикулума из 1994. године, и једна од најважнијих карактеристика новог курикулума је редукција такозваних тема које нису битне и основне. Можемо да закључимо да су индонежанске математичке курикуларне реформе умногоме биле под утицајем и у складу са трендовима других земаља, и следећа два става су нарочито битна. Прво, није постојао оквир националног математичког курикулума који је водио земљу у реформисање курикулума. Друго, постојала је врло слаба процена потреба у реформисању курикулума у прошлости, и наше мишљење је да начин којим се води процена потреба у одређеном образовном, економском и друштвеном контексту питање које треба да буде врло битно за математичке курикуларне реформе и развој, како у Индонезији, тако и у другим земљама.

Кључне речи: историја математичког образовања, индонежанско математичко образовање, математичка курикуларна реформа и развој.

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Atsumi Ueda, MEd
Hiroshima University, Graduate School of Education, Hiroshima, Japan

Takuya Baba, PhD
Hiroshima University, Graduate School of Education, Hiroshima, Japan

Taketo Matsuura, MEd
Hiroshima University, Graduate School of Education, Hiroshima, Japan

Values in Japanese Mathematics Education from the Perspective of Open-ended Approach

Abstract: *Mathematics education community in Japan has continuously and extensively developed 'mathematical thinking' as an educational value. In this paper, the historical review was conducted on mathematical thinking in terms of its evaluation and educational method, textbook change, and research on treatment of diversified mathematical thinking. This approach can provide methodologically an important perspective to grasp, clarify and make relative the values in mathematics education in different times of each culture. Values here mean those attitudes which lay at the back of the intention, judgment, and selection of teaching-learning activity exhibited by primary teachers. As a result of this research, it is learnt that the theme in mathematics education research does reflect values held by the primary mathematics teachers. They, in turn, have held central ideas and value utilizing children's diversified mathematical thinking, letting them subjectively and extensively construct mathematical ideas in the lesson. The major characteristics of Japanese Mathematics education is the open-ended approach, which has been developed as an evaluation and educational method of mathematical thinking. This is available as translated version of "The Open-Ended Approach: A New Proposal for Teaching Mathematics" (The original version (Shimada) is in Japanese published in 1977).*

Keywords: *Value, Open-ended approach, Historical analysis, Mathematical Thinking.*

Introduction

The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom (Stigler & Hiebert, 1999) called in earnest for attention to be paid to the lesson study and Mathematics education in Japan, where the lesson study has been developed. In the same vein, a few other international efforts sought to introduce Japanese Math-

1 takuba@hiroshima-u.ac.jp

ematics education to the international community, and one of the typical ones is EARCOME 5 (East Asian Regional Conference on Mathematics Education). Such efforts are rooted in the regional characteristic of Japan however, and they tend to be bound by Japanese perspective. The two mentioned initiatives however, start to raise fundamental questions over what introducing the Mathematics education in a particular country such as Japan means, what is first of all the Mathematics education in Japan like, and what has been valued in Japanese Mathematics education by many people who have been involved in it. In return, these questions create the necessity of reflecting and giving some answers.

In 2006, JASME (Japan Academic Society of Mathematics Education) held the symposium during the 22nd annual conference with the theme of Cultural Aspects of Mathematics Education in Japan with a focus on Mathematical Thinking. It aimed at grasping and describing 'mathematical thinking' as an educational principle, which mathematics education community in Japan has continuously and extensively valued and developed, and exploring the future direction of it through the reflection on its characteristics. The symposium confirmed that the whole clothe of mathematics education in Japan has developed coherently with mathematical thinking being as it were its warp and social and historical needs as its weft (Baba, 2006).

In this paper, the persisting values of mathematics education community in Japan are reflected from teachers' perspective as an example of open-ended approach. Here the values mean "those of primary teachers at the back of their intention, judgment, and selection of teaching-learning activity" (Baba et al., 2013). The open-ended approach is a good example in order to relativise and reflect characteristics of Japanese mathematics education, because it retains those characteristics developing around mathematical thinking and diverse ideas, and it can be also referred by international researchers since it is available as translated version "The Open-ended Approach: A New Proposal for Teaching Mathematics" (Becker & Shimada) published in 1997.

The open-ended approach is typically exemplified as the developmental work with *Open-ended approach in Mathematics Education - New Proposal of Lesson Improvement* (Shimada, 1977) and *From Problem to Problem –Extensive Treatment of Problems for Improvement of Mathematics Lesson* (Takeuchi & Sawada, 1984). This extensive treatment of mathematical problems is seen as an extension of Open-ended approach, and thus it is included in it.

Emergence of Mathematical Thinking as the Objective in the Course of Study

The term "mathematical thinking", which is a translation of "suugakutekina-kangaekata", first appeared in 1958 in the objective of the course of study for primary education. The course of study was developed in Japan after the WWII and was intended to be the national curriculum in Japan. Its objectives at the time were as follows:

1. To enable students to understand basic concepts and principles about numbers and quantities, and geometrical figures, and let them develop more advanced <u>mathematical thinking and how to treat it.</u>
2. To enable students to acquire basic knowledge and fundamental skills about numbers and quantities, and geometrical figures, and let them use those effectively and efficiently according to the purpose.
3. To enable students to understand the significance of using mathematical terms and symbols, and let them use expression and think simply and clearly quantitative events and relations using the terms and symbols.

4. To enable students to extend the abilities to set up a appropriate plan and to think logically regarding quantitative events and relations, and let them treat things more self-dependently and rationally.
5. To enable students to develop attitudes towards a proactive mathematical thinking and how to treat it in daily life. (Underlined by the authors.)

The phrase "mathematical thinking and how to treat it" in this objective is commonly referred as "mathematical thinking" to mean all components related to this mathematical thinking and treatment. From the above, it is expected to develop the acquired fundamental concepts and basic skills to the more advanced level and to grow the attitudes to apply them extensively to daily life situation. Historically speaking, the mathematical idea[2] as philosophical stance in national textbook *Jinjo-shogaku-sanjutsu* used since 1935 preceded the mention of mathematical thinking (Ueda, 2006). So in this sense, there was a continuing aspiration of Japanese mathematics education community despite of temporal mutation during the WWII.

The community at the time tried to uplift the lowering standards of mathematics education when the term 'mathematical thinking' emerged, after the critical reflection over the life unit learning which placed mathematics education as skills-based subject (Nakashima, 1981). In other words, the community aimed at raising efficiency by teachers' clarifying and extending the basic ideas and principles through mathematical thinking. Through development of mathematical thinking abilities, students would have been able to find out new ideas subjectively and use appropriately and efficiently mathematical facts and relations, express and think of them in a concise and clear way, and treat them independently and rationally. Despite these intentions, the meaning of mathematical thinking at the time was not clear enough to the majority of teachers expected to teach it.

Just before the emergence of the concept of mathematical thinking, there was a preceding idea, called the central concepts. The term first appeared in the course of study for the senior high school in 1956. The characteristics of this course of study were the integration of Analysis I, Analysis II and Geometry into Mathematics I, Mathematics II and Mathematics III as mathematical subjects. At that time, central concepts exemplified mathematical thinking as central ideas to bridge all the content of each mathematical subject although they were shown separately in terms of the algebraic and geometrical contents. For example, the central ideas for Mathematics I were described as follows:

a. Expressing the concepts in symbols
b. Extending concepts and laws
c. Systematizing knowledge by deductive reasoning
d. Grasping relation of correspondence and dependence
e. Finding out invariance of equation and geometrical figures
f. (Identifying) Relations between analytical and geometrical methods.

The central ideas had an intention to integrate algebra and geometry in mathematics as one subject and to extract mathematical methods and activities common to both of them. They are not exactly the same as mathematical thinking, which has become an objective of primary mathematics education, but it certainly had an influence on its introduction. When the course of study was revised in 1968 to introduce the idea of modern mathematics movement, it further emphasized the mathematical thinking we have been talking about.

The table 1 shows the name of sessions and the number of presentations in the session during

2 Mathematical ideas are philosophical attitudes to love and enjoy mathematical philosophy in pursuit and acquisition of mathematical truth, and to find and consider the mathematical relation in the daily events and to take an action based on them (Shiono, 1970).

the annual conference by the Japan Society of Mathematics Education (JSME). When the course of study was revised in 1968, the sessions on the newly introduced topics such as sets, function and probability and statistics were created in addition to the existing ones such as number and calculation, quantity and measurement, geometrical figures. The session of mathematical thinking was created only 6 years later in 1973. In other words, discussion over mathematical thinking started after discussion over the above contents had reached a certain level.

Efforts analyzing and defining the mathematical thinking

Around the time of setting the session at the JSME in 1973, the analysis on concepts of mathematical thinking had already started. Katagiri of Tokyo Metropolitan Institute of Education ushered in the concept into his analysis *Mathematical Thinking and its Teaching* (Katagiri et al.) in 1971 and categorized mathematical thinking into three types. These were identified as

a) the attitudinal aspects of mathematical thinking

b) the process aspects of mathematical thinking such as generalization and analogy, and

c) the contents related mathematical thinking such as unitary amount and relative amount.

In 1981 Nakashima published *Mathematical Thinking at Primary and Secondary Mathematics Education*, and stated that mathematical thinking consisted of abilities and attitudes to work autonomously and have an ability to apply these creatively through an activity appropriate to mathematics education. He clarified that to develop mathematical thinking, one had to pay attention to this autonomous and creative process within an activity.

In 1988, Katagiri reorganized the above categorization of mathematical thinking into mathematical thinking related to methods and contents. Through these publications, interpretation of mathematical thinking has been gradually clarified in Japanese context. As we have seen so far, Katagiri and Nakajima have been the most famous researchers that contributed to analytical research on mathematical thinking in Japan.

Table1. Sessions and Number of Presentations at Annual conference of JSME (Primary School)

Sessions \ Year	…	1965	1966	1967	1968	1969	1970	1971	1972	1973	1974	1975	1976	1977
Number and calculation	…	9	15	23	4	13	25	16	19	21	10	32	37	47
Quantity and measurement	…	8	12	12		11	8	5	2	10	9	6	7	9
Geometrical figures	…	7	17	17	2	16	18	10	17	12	19	25	33	26
Problem solving	…	20	33	27	8	12	9	5	6	9	3	6	8	11
Sets					25	25	13	14	11	9	2	7	4	5
Function					31	18	19	15	18	17	16	20	26	11
Probability and statistics					6	21	13	14	13	15	8	14	9	7
Mathematical thinking											14	15	7	14

As shown in the following section, the research on evaluation and concretization of mathematical thinking has been developed simultaneously, while the above type of analysis continued. Both of these approaches – the analytical research and the concretization – we see as the different sides of the same coin, and they have been influencing and referring to each other and deepening as a whole the field of enquiry related to mathematical thinking.

Evaluating and developing the mathematical thinking through the Open-ended approach

For six years between 1971 and 1977, Mathematics education researchers in NIER (National Institute of Education Research): university professors, primary and secondary school teachers, formed an interest group and developed the research project, whose theme was to develop evaluation method of mathematical thinking, through the Grant-in-Aid by the Ministry of Education, Culture and Sports. This group, consisting of about 30 members, scrutinized the objectives of primary mathematics education carefully and stated that "mathematical thinking has been flowing at the bottom of mathematics education in Japan since mathematical ideas in *Jinjo-shogaku-sanjutsu* (the national textbook during the pre-war period, Grade 1 of which was published by the Ministry in 1935) aiming to develop mathematical and scientific thinking, and the course of study in primary and junior secondary schools has already clearly stated it in 1958 and in senior secondary school in 1956. … In short, to be able to develop mathematical thinking can be regarded as the ultimate goal of mathematics education" (Hasihmoto, 1976: 21-22).

This interest group further conducted the survey questioning a wide array of stakeholders, from mathematicians, mathematics education researchers, to mathematics teachers across Japan, regarding some behavioral examples of primary and secondary students attaining the objectives of mathematical thinking. The group summarized the findings from the answers about mathematical thinking as containing the following items (Hashimoto, 1976: 22), that are not necessarily independent from each other:

1. Being able to find out relations that underline the situation within a problem and begin to construct it mathematically.
2. Being able to solve non-routine problems which cannot be solved by common procedures.
3. Being able to develop something new.
4. Being able to fulfill one's own ideas in the group.
5. General objectives (under the current course of study).

Following these findings, the researchers repeated the process of developing the Open-ended problems for evaluation and trialed them in classroom. They had hypothesis that attainment level of mathematical thinking can be assessed through such incomplete Open-ended problems. They used these problems in the lessons and let students find out as many relations as possible and describe mathematically those relations. Evaluation is done by analyzing the relations in terms of quantity and quality, which is sophistication level of their description (Sawada & Hashimoto, 1972: 65).

The notable point for this project is that it focuses not only on evaluation method but also on effective teaching strategies to realize development of mathematical thinking. This basic stance of the group influenced the direction of the research.

The research theme for the first year following the research project was "development research on evaluation method in mathematics education" but it was changed for the second and third year into "development of evaluation method in mathematics education and analysis of impact of various factors". Analysis of the factors is made possible by the fact that data collection at the classroom level had been done intensively from the beginning of this development work. In fact, the work done in the project paid attention to the students' group discussions during the lesson and tried to evaluate the change of this group discussion for the second year (Sawada & Hashimoto, 1972).

The experience and knowledge gained through the research project, and which have been accumulated through the data collection regarding students' responses, prompted the group to shift from "development work of evaluation method for mathematical thinking" to "development of teaching strategies for mathematical thinking." Even after this, students' responses in the lessons had been continuously collected in parallel to sophistication of evaluation method. And gradually they become a new standard of teaching strategy.

The report for fifth year stated that the objective was "This year it aimed at trialing a few incomplete open-ended problems in the lesson during the second semester and confirming through statistical survey if this form of teaching can promote the attainment of the above objectives, and showed also the following results from teachers' observation and students' remarks" (Shimada, 1976, 29-30):

a. The middle and low achievers with less activity have become more active in expressing their ideas. (It is the same as the previous year).

b. Especially the middle achievers in the daily activity have made most remarkable progress in elementary and junior secondary schools. (It is the same as the previous year).

c. Some of the high achievers in senior high schools have performed less than previous year, since they become too careful not to make a mistake. (It is the same as the previous year).

d. In the previous year, it was reported in elementary schools that there were a few students who showed interests in mathematical properties after finding them within an open-ended problem, but in this year there were many examples which showed students took interest in these properties.

In the same year, the following ideas about teachers' work were found to be the case.

e. It may not be possible to say that being incomplete makes the problem effective. Rather, a problem posed to students should not only be incomplete, but it should also have a certain direction towards a solution, and something that is produced by students while they work on it, should be mathematically significant.

f. The open-ended problem approach is effective both at the introduction and at the summary of the lesson. When there is a good problem at the introduction, the lesson development becomes interesting. When it is given at the end or while summarizing the lesson, it is useful to review various aspects learned.

As for the summary and future issues of the research, the two points were listed as follows.

The first point is that the two terms of the year during which the research project took place were too short to confirm effectiveness of the teaching approach based on open-ended problems. Changes that were expected would be more visible only after a longer time has been spent in dedicating time to mathematical thinking and open-ended problem solving in the classrooms. In this sense, it was recommended to plan the activity from the beginning of school year in the following year.

The second point is that the problems used in the lessons were diverse not only in results but also in the processes and contexts they represented. Consequently, they had given diverse results, which could not always be correlated or compared with each other.

And you can see in the above point, diversities were noted in the process of research on evaluation method of mathematical thinking, and they demanded the necessity of systematizing and theorizing them as mathematical activities. "Problem posing with diversity" was used as the evaluation method on development of mathematical thinking. In other words, mathematization of phenomenon was placed in the center of mathematical thinking, and it was assumed that possibility of such mathematization was not only one but also several, and so the significance of "being diverse" was to be re-considered. It is notable that the research group located this as the future issue.

This research resulted in perceiving mathematical activity as coming and going between real and mathematical world and locating it in the phases of the Open-ended approach, which extensively utilizes the incomplete problems (Figure 1). And Takeuchi employed theory of scientific knowledge growth by Popper and approached this issue from the perspective of the nature of mathematical activity (Takeuchi, 1976: 11-12). This consideration played an important role in shifting the research from the Open-ended approach to "extensive treatment of problems" (Takeuchi, 1984: 9-23).[3]

In this way, the developmental research of the Open-ended approach has continued as mutual interaction between theory and practice and the treatment of diversity in mathematical learning has become systematized.

Diversified ideas in mathematics textbooks

Lesson development like the one described above, and which used diversity of mathematical ideas has made an impact also on lesson structure. The textbooks published by one textbook company were compared by focusing on the area of plane figure (parallelogram) in the fifth grade. The textbooks from 1965 to 1975 (Figure 2) didn't have diversified ideas, but they has already started development of evaluation method of mathematical thinking using incomplete problems. The textbooks in 1980 (Figure 3) and that in 1985 (Figure 4) contained more than one idea. They are different ways of "cutting into pieces and pasting them together" and "moving trapezium and matching the corresponded areas". We must remember that the latter took place after the Open-ended approach was proposed as teaching method in 1977. This leads to current textbook (Figure 5). Adoption of diversified ideas in the textbook produces the practical issues on how to treat them during the lesson.

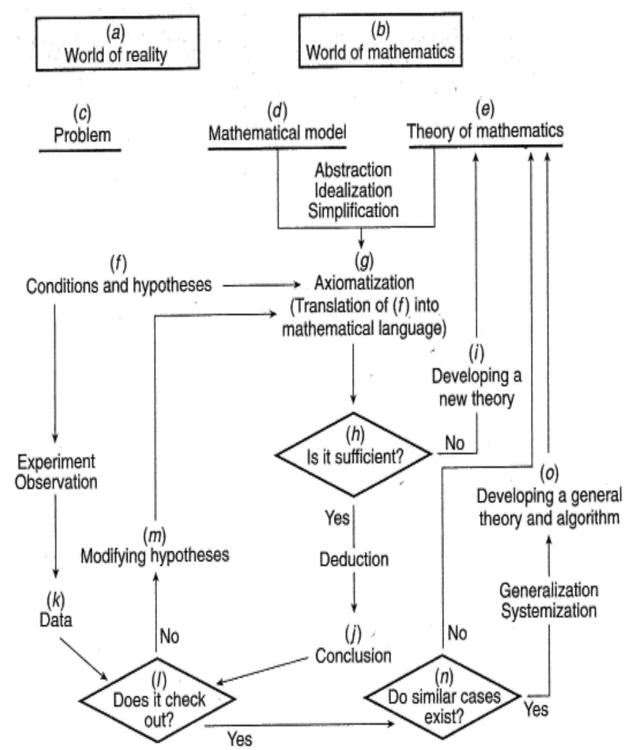
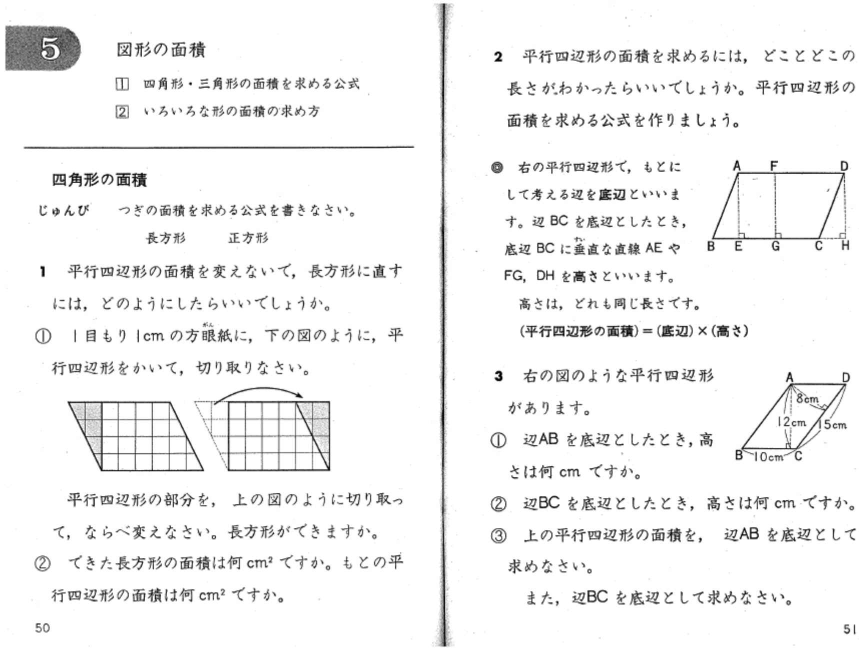


Figure 1. Model of Mathematical Activity (Shimada, 1977: 15)

3 Starting from one problem given to the children, which is called a original problem, children are encouraged to pose new problems through replacing the component of the original problem with similar and more general ones and considering the converse, and to develop subjective attitudes to solve the problems for themselves (Takeuchi, Sawada, 1984: 25).



5 図形の面積

1 四角形・三角形の面積を求める公式

2 いろいろな形の面積の求め方

四角形の面積

じゅんび　つぎの面積を求める公式を書きなさい。

長方形　正方形

1 平行四辺形の面積を変えないで，長方形に直すには，どのようにしたらいいでしょうか。

① 1目もり1cmの方眼紙に，下の図のように，平行四辺形をかいて，切り取りなさい。

平行四辺形の部分を，上の図のように切り取って，ならべ変えなさい。長方形ができますか。

② できた長方形の面積は何cm²ですか。もとの平行四辺形の面積は何cm²ですか。

50

2 平行四辺形の面積を求めるには，どことどこの長さがわかったらいいでしょうか。平行四辺形の面積を求める公式を作りましょう。

◎ 右の平行四辺形で，もとにして考える辺を底辺といいます。辺BCを底辺としたとき，底辺BCに垂直な直線AEやFG，DHを高さといいます。

高さは，どれも同じ長さです。

(平行四辺形の面積)＝(底辺)×(高さ)

3 右の図のような平行四辺形があります。

① 辺ABを底辺としたとき，高さは何cmですか。

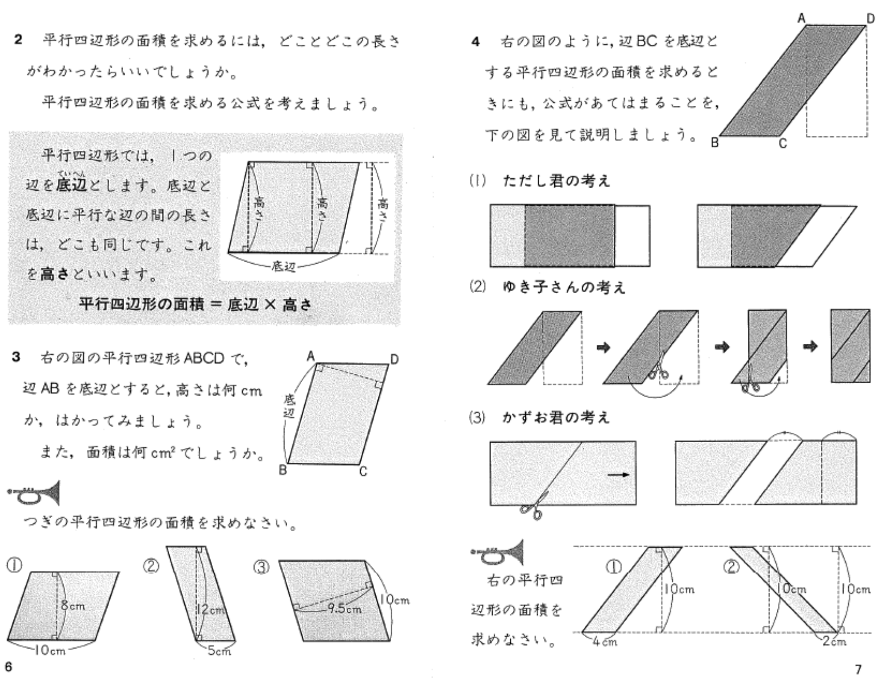
② 辺BCを底辺としたとき，高さは何cmですか。

③ 上の平行四辺形の面積を，辺ABを底辺として求めなさい。

また，辺BCを底辺として求めなさい。

51

Figure 2. Mathematics textbook in 1974



2 平行四辺形の面積を求めるには，どことどこの長さがわかったらいいでしょうか。

平行四辺形の面積を求める公式を考えましょう。

平行四辺形では，1つの辺を底辺とします。底辺と底辺に平行な辺の間の長さは，どこも同じです。これを高さといいます。

平行四辺形の面積＝底辺×高さ

3 右の図の平行四辺形ABCDで，辺ABを底辺とすると，高さは何cmか，はかってみましょう。

また，面積は何cm²でしょうか。

つぎの平行四辺形の面積を求めなさい。

6

4 右の図のように，辺BCを底辺とする平行四辺形の面積を求めるときにも，公式があてはまることを，下の図を見て説明しましょう。

(1) ただし君の考え

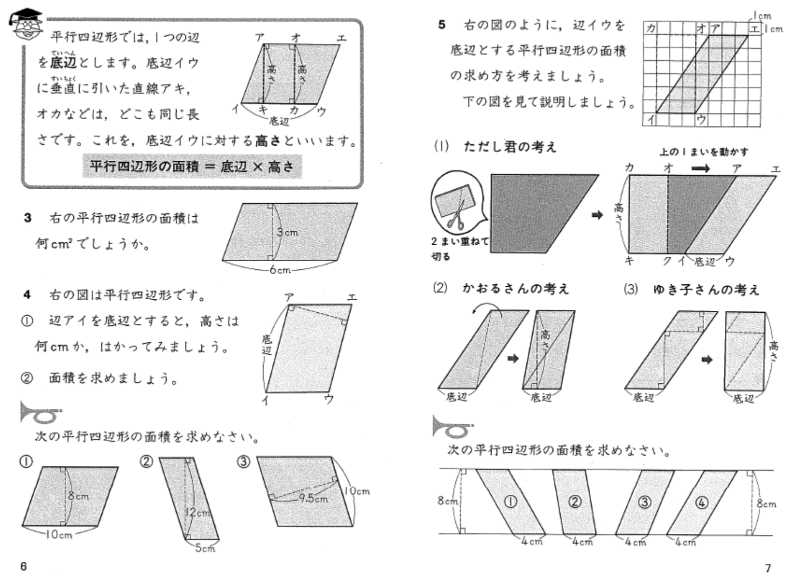
(2) ゆき子さんの考え

(3) かずお君の考え

右の平行四辺形の面積を求めなさい。

7

Figure 3. Mathematics textbook in 1980


平行四辺形では，１つの辺を底辺とします。底辺イウに垂直に引いた直線アキ，オカなどは，どこも同じ長さです。これを，底辺イウに対する高さといいます。

平行四辺形の面積 ＝ 底辺 × 高さ

3 右の平行四辺形の面積は何cm²でしょうか。

4 右の図は平行四辺形です。

① 辺アイを底辺とすると，高さは何cmか，はかってみましょう。

② 面積を求めましょう。

次の平行四辺形の面積を求めなさい。

6

5 右の図のように，辺イウを底辺とする平行四辺形の面積の求め方を考えましょう。

下の図を見て説明しましょう。

(1) ただし君の考え

2まい重ねて切る

上の１まいを動かす

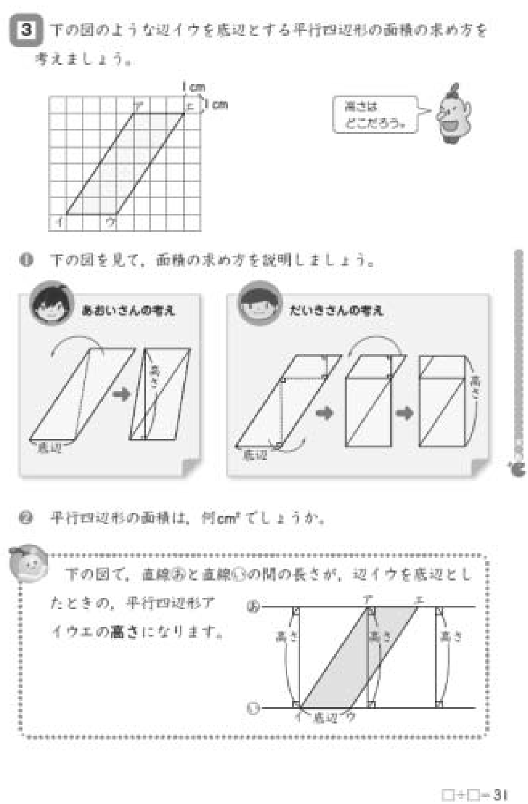
(2) かおるさんの考え

(3) ゆき子さんの考え

次の平行四辺形の面積を求めなさい。

7

Figure 4. Mathematics textbook in 1985


3 下の図のような辺イウを底辺とする平行四辺形の面積の求め方を考えましょう。

高さはどこだろう。

❶ 下の図を見て，面積の求め方を説明しましょう。

あおいさんの考え

だいきさんの考え

❷ 平行四辺形の面積は，何cm²でしょうか。

下の図で，直線あと直線いの間の長さが，辺イウを底辺としたときの，平行四辺形アイウエの高さになります。

□+□=31

Figure 5. Mathematics textbook 2010

Research on how to treat and summarize diversified ideas

The diversification of ideas as the ones shown above which made their way into the Japanese textbooks have influenced the developmental research on how children treat and summarize different mathematical ideas during the learning process. "One objective of the problem solving through diversified ideas is to ensure acquisition of the basic knowledge and skills and the understanding of mathematical thinking which can be encountered in the process of learning through presentation of those ideas, and to aim at the development of individual student's holistic growth including cognitive understand, emotional development and explaining skills through the whole class participation" (Koto, 1992, 19). Koto further stated that diversified ideas should lead to development of mathematical thinking.

Koto (1992, 1998) classified diversified ideas, which can be observed during mathematics lesson, in terms of teaching aims and quality, and proposed the instruction flow utilizing them as follows:

Independent diversity: Paying attention to validity of each idea

Prioritized diversity: Paying attention to efficiency of each idea

Integrated diversity: Paying attention to commonality of each idea

Structured diversity: Paying attention to mutual relations between ideas.

The research on how to treat and summarize the diversified ideas is regarded as one of the necessary items for the lesson study in Japan in the spe-

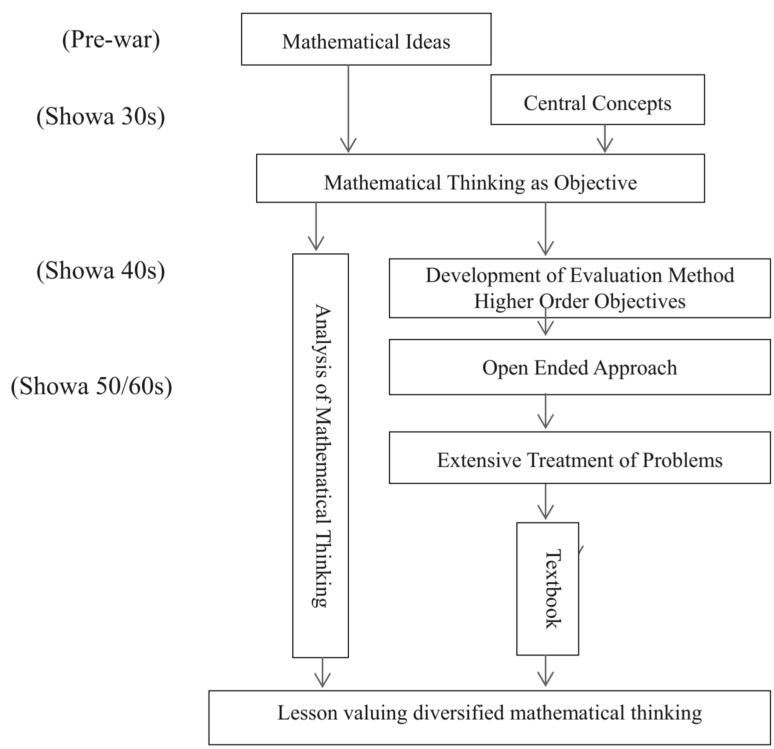


Figure 6. Flow of Mathematics Education in Japan from the Perspective of Open-ended Approach

cial issue *Theory of mathematics education in Japan for lesson study*, which was published by Japan Society of Mathematics Education (JSME) during EARCOME in 2010 (Wada, 2010). This shows the significance of research impact by Koto and others on the lesson development in Japan.

Summary

Engagement by Japanese mathematics education community regarding the open-ended approach can be summarized chronologically in the Figure 6.

Most primary mathematics teachers let the students construct mathematical ideas subjectively and extensively, and valued utilizing children's diversified mathematical ideas in the lesson. In this paper, the historical development was reviewed by taking up analytical and developmental researches on mathematical thinking. The latter described development research has changed its focus from evaluating the method of mathematical thinking through the open-ended approach to treatment of diversified mathematical ideas. The described history of transition from mathematical thinking, through open-ended problems to diversified ideas, shows the forming of shared values among the Japanese mathematics education community. Historical approach regarding research method, research themes and developmental processes of research can provide methodologically an important perspective to help us grasp values in mathematics education in each culture and society, not only Japan.

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мр Ацуми Уеда
Педагошки факултет, Универзитет у Хирошими, Јапан
др Такуја Баба
Педагошки факултет, Универзитет у Хирошими, Јапан
мр Такето Мацура
Педагошки факултет, Универзитет у Хирошими, Јапан

Вредности јапанског математичког образовања из перспективе „отвореног приступа“

Књига „Јаз у настави“ (Stigler & Hiebert, 1999) привукла је пажњу стручне јавности представљањем међународној заједници „студије часа“ и јапанског математичког образовања, посебно дискутујући о „студији часа“. Са друге стране, асоцијација JASME (Japan Academic Society of Mathematics Education) одржала је симпозијум током 22. годишње конференције на тему културног аспекта у математичком образовању у Јапану. Да је „студија часа“ континуирано и екстензивно развијала „математичко мишљење“ као образовну вештину, истражујући и њен будући правац развоја кроз саморефлексију њених карактеристика, потврдио је Баба (Baba, 2006). У овом раду се на математичко мишљење историјски гледало више из перспективе наставника у основној школи, у смислу евалуације и метода рада, промена насталим у уџбеницима и истраживању различитих математичких идеја. „Отворени приступ“ је узет као пример, јер се односи на све наведене аспекте. „Математичке идеје“ као филозофско питање у националном уџбенику Jinjo-shogaku-sanjutsu се користе од 1935. године (Ueda, 2006). И пре појаве термина „математичко мишљење“ у настави средње школе још од 1956. године постојао је термин „централни појам“, са намером да се издвоје математичке методе и активности заједничке алгебри и геометрији и да се интегришу у један предмет. Наведени термин „централни појам“ није био истоветан термину „математичко мишљење“, али је сигурно утицао на његово увођење. Онда се нов термин појавио 1958. године као циљ курса основношколског образовања. И кроз прихватање математичког мишљења, од ученика се очекивало да досегну нове идеје самостално и да користе математичке чињенице и односе међу њима смислено и ефикасно, да их изражавају и да промишљају о њима на концизан начин, и да тачно поступају са њима, независно и рационално. Упркос свим напорима, значење новог термина у то време није било јасно. Катагири (Katagiri et al., 1971; Katagiri, 1988) анализирао је значење и категоризовао математичко мишљење. Накашима (Nakashima, 1981) математичко мишљење схватао је као способност самосталног рада и остао је при идеји да је то аутономни и креативни процес. Кроз рад Катагирија и Накашиме, значење новог термина постало је јасније. Од 1971. године, па наредних шест година, истраживачи у Институту NIER (National Institute of Education Research), професори универзитета и наставници у основним и средњим школама формирали су интересну групу и развили истраживачки пројекат чија је тема била развијање евалуационог метода математичког мишљења, који је касније назван „отвореним приступом“ (Shimada, 1977). Пројекат је користио активно незавршене проблемске ситуације, које су стварале разноликост не само по резултатима већ и у самом процесу и контекстима. Искуство и знање које су ученици стицали у пројекту акумулирани су кроз скупљање података који су се односили на одговоре ученика и чинили да се пројекат развије од „евалуационог модела за математичко мишљење“ до „наставне стратегије за математичко мишљење“. Није довољно имати различите идеје шта деца могу, већ треба потврдити да ове идеје имају образовну вредност. Да би се утврдиле овакве идеје, неопходно је организовати смислене математичке активности у теорији. То нас је довело до схватања математичке идеје као нечег између стварног и математичког света, што је имало за сврху различите идеје које се појављују при решавању незавршених проблема. Такве идеје класификовао је Кото (Koto, 1992, 1998) у терминима циљева и квалитета учења и предложио је постојање инструкција за учење. Истраживање је имало огроман утицај на развијање система часова у Јапану. Коначно, откривено је да је „отворени приступ“ остао основна карактеристика јапанског математичког образовања. Анализа значења „математичког мишљења“, развој евалуације, те развој самог термина сигурно су били у међусобној интеракцији и развијали се као целина. Претпоставка је да су сви они имали огроман утицај на укупну вредност математичког образовања у Јапану. А историјска анализа може да омогући једном методичком приступу да појасни и релативизује вредности математичког образовања у различитим временима сваке културе.

***Кључне речи*:** вредност, „отворени приступ“, историјска анализа, „математичко мишљење“.

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Karmelita Pjanić[1]**, PhD**
Faculty of Educational Studies, University of Bihać, Bihać, Bosnia and Herzegovina

Original Paper

The Origins and Products of Japanese Lesson Study

Abstract: *The aim of this paper is to provide an overview of the Lesson Study and its main product – Problem Solving Approach, based on the relevant literature research and direct observations by the author of the paper. Japanese Lesson Study is recognized as successful methodology in Mathematics education. In the Western European countries and the United States, Lesson Study is usually perceived as a professional development process that engages Japanese teachers to systematically examine their practice with the goal of becoming more effective. However, Lesson Study is more than professional development. It is a scientific activity for teachers based on methodology introduced in 1880s (Isoda, 2011). The products of Lesson Study are not limited only to what participants had learned from particular class and post-class reflective discussion. It also includes the development of local theories in Mathematics education. One of theories of teaching Mathematics that emerged from Lesson Study is Problem Solving Approach, which is commonly known as Japanese teaching approach and theory of teaching about learning how to learn (Stigler & Hiebert, 1999). The aimed product of this approach is the ability of students to learn mathematics independently.*

Keywords: *lesson study, problem-solving approach, Mathematical education.*

Introduction

International assessments in Mathematics reveal consistent high performance of students from Japan and other Asia Pacific countries. This puts educational systems and practices of Asia Pacific countries into focus of interest for the Mathematics education community as well as curriculum policy, design and development community. One of the topics of worldwide attention is the Japanese Lesson Study, which is recognised as an engine for above-average achievement of Japanese students. Lesson Study is a process where teachers collaboratively develop ways to foster students' what may some call 'flexible' understanding of Mathematics. Lesson Study has been used in Japan since 1880's with the purpose of improving the preparation of a lesson, sequence of lessons or a selected topic, to predict pupils' reaction, and to review and improve a lesson studied in the cycle of improvement. Generally, the lessons are based on a problem-solving approach in which teachers educate pupils to think for themselves. In USA and some European countries Lesson Study is usually perceived as a professional development process that engages teachers to systematical-

1 kpjanic@gmail.com

ly examine their practice with the goal of becoming more effective. However, Lesson Study is more than the professional development. The aim of this paper is to provide an overview of the Lesson Study and its main product – Problem Solving Approach, based on relevant literature research and direct observations by the author of the paper.

Meaning and origin of Lesson Study

Lesson Study is a process in which teachers progressively and systematically strive to improve their teaching methods by working with other teachers to examine and critique each others' teaching techniques. This examination centers on teachers working collaboratively on a number of "study lessons". Working on study lessons involves planning, teaching, observing, reflecting and critiquing the lessons. To provide focus and direction to this work, the teachers select an overarching goal and related research question that they want to explore. The research question then serves to guide their work on all aspects of the lessons they study. The origins of Lesson Study can be traced to educational practice in the Meiji period of Japan. It began from the observation of teaching methods in whole classroom teaching which had been firstly introduced in schools beyond the temple school culture which used tutorial teaching methods. According to Wakabayashi and Shirai (cited in Isoda, 2011), Lesson Study first began at the Tokyo Normal School, which later became the University of Tsukuba, in 1870s. From the very begining teachers were focused on argumentation through questioning instead on lecture style method. People observed the ways of teaching for knowing how-to conduct teaching and learning process (Makinae, 2010). As a result, Teachers' Canon was published by the Normal School in 1873, which described the etiquette for entering classroom, for observation of lessons, and for avoiding the negative effects of observations (Inprasihta, 2006). Since then Lesson Study has functioned in Japan as a way of enabling teachers to develop and study their own teaching practices and shed light on the local theories of education.

Derived from the Japanese words *jugyo kenkyuu*, the term *lesson study* was coined in 1999 by Makoto Yoshida in his doctoral dissertation *Lesson study: A case study of a Japanese approach to improving instruction through school-based teacher development* (Takahashi & Yoshida, 2004). It can also be translated and interpreted as *research lesson*.

Generally, the following four dimensions of Lesson Study can be pointed out:

- The collaborative activity,
- The form of research related to lesson,
- Importance that the pupils are kept at the heart of the process,
- The understanding of the process that is primarily focused on content and pupils rather then on technology and tools.

The cycle of the lesson study goes as follows. While working on a study lesson, teachers jointly draw up a detailed plan for the lesson, which one of the teachers uses to teach the lesson in an actual classroom and other members of the group observe. This group of observes may be joined by others: observers could be just the faculty within a school, or a wider group: teachers from several schools sometimes joined the university instructors and supervisors from the board of education. Usually observers record lessons, as well as their impressions of lessons in multiple ways. A discussion of the lesson follows. Typically, such a gathering begins with presentations by the teachers who taught and co-planned the lesson, followed by free or structured discussion. Upon review of the lesson, another teacher usually implements it in a second classroom, while group members again observe. The group will come together again to discuss the observed instruction. Finally, the teachers produce a report of what their study lessons have taught them, particularly with respect to their research question.

Research lessons are designed to bring to life in a lesson a particular goal or vision of education. The whole faculty chooses a research theme or focus. Typically, it is a broad goal or vision of education that goes beyond a specific subject matter and lesson. Table 1 shows samples of Lesson Study Topics with broad goals which were implemented in Japan on national level. (Isoda, 2011).

Table 1. Lesson Study Topics

Period	Lesson Study topic
1880s	Pestalozzi Method and Dialog Method (including argumentation/ discussion/dialogue between teacher and students)
1910s	Mathematics for Life (including problem posing)
1930s	Curriculum Integration in Mathematics (including Open-Ended Problems)
1950s	Core curriculum movement based on social studies
1960s	Mathematical Thinking (Japanese way of New Math)
1970s	Open-Ended Approach and Problem Solving Approach
1980s	Problem Solving

The table allows us to follow the chronology of changes in the Lesson Study topics and approaches for developing children in Japan during one century.

Japanese Lesson Study features

Japanese Lesson Study is recognized as having the following elements (Isoda, 2011): determined process/cycle, open classroom[2], theme or focus for Lesson Study, lesson plan, teachers acting as researches, results, sequential experience for sharing the heritage and development of children who learn by/for themselves. We will take a closer look into these features.

1. In short, the process of Lesson Study can be described as process: *plan, do and see*. The activities related to 'plan, do, and see' are conducted collaboratively by teachers and repeated in cycles. The first stage of the Lesson Study process is preparation or goal-setting and planning (*kyozai kenkyu*). This process begins with finding and selecting materials relevant to the purpose of the lesson. It is followed by joined work of teachers in refining the lesson design based on the actual needs of the pupils and tying all of this information together into a lesson plan. Based on the joint teaching plan, teacher conduct a lesson in an open classroom while the group members and outsiders observe the class, taking detailed notes regarding the reactions and engagement of the students. This represents second stage of cycle – teaching the research lesson enables the lesson observation (*koukai/kenkyu jugyo*). Review session (*jugyo kentoukai*) is held for all observers after the research lesson. The group comes together to discuss and reflect on the instruction witnessed and what it taught them about the goal they set out to explore.

The reflective nature of Lesson Study has as its premise the collaboration between participants, and throughout the process emphasis is placed on how pupils view and comprehend the subject matter being taught (Sarkar at al., 2010). The methodology inherent in conducting Lesson Study leads to the need for effective documentation of classroom observations. One of such documents is presented in Table 2. It is primarily designed for self-evaluation, however it could be used by all participants in a lesson study group.

2 Term „open classrom" refers to allowing other teachers, students and educational experts to observe lessons.

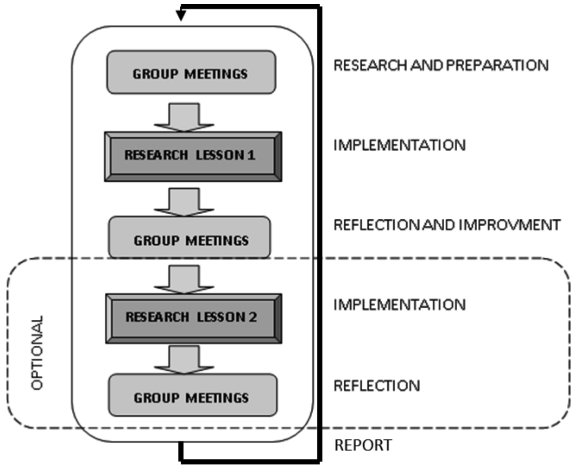

Figure 1. Typical Lesson Study cycle

Table 2. Lesson Planning Checklist: Self-Evaluation (addapted from Isoda and Olfos, 2009)

Problem Posing	
1. The lesson sets tasks that can be solved in a variety of different ways by applying previously learned knowledge,	
and presents the content to be learned.	4 3 2 1
2. The lesson is planned with tasks (problem given by teacher) and problems (problematic from students), and pro-	
motes problem (problematic) awareness.	4 3 2 1
3. The teacher anticipated the methods and solutions.	4 3 2 1
Independent Solving	
1. The children can recall and apply what they have already learned.	4 3 2 1
2. The children's ideas are anticipated.	4 3 2 1
3. Inappropriate solutions are predicted, and advice and hints are prepared in advance.	4 3 2 1
4. The teacher, walks around, observes and helps children to ensure that children use mathematical representation	
to solve the problems.	4 3 2 1
5. Notes are written in a manner such that they will be helpful for presentation as well.	4 3 2 1
Comparison and Discussion	
1. Steps (Validity, Compare, Similarity and Generalisation or Selection) are planned for comparative discussion.	4 3 2 1
2. The ideas to be taken up are presented in an order that is planned in advance.	
3. The method for writing presentation sheets is planned in advance and directions are	4 3 2 1
provided.	4 3 2 1
4. In addition to developing the ability to explain, children also foster the ability to listen and to question.	
5. When ideas are brought together (generalised), it is important to experience them by	4 3 2 1
themselves.	4 3 2 1
6. The reorganisation or integration of ideas proceeds smoothly from the presentation	
and the children's communication.	4 3 2 1
Summary	
1. Activities are incorporated that let children experience for themselves the merits of	4 3 2 1
the ideas and procedures that are generalised.	
2. The summary matches the aims and problems (problematic) of this lesson.	4 3 2 1
3. It is recognised that both correct and incorrect answers (to the task) are good in the	4 3 2 1
formation of their ideas.	
4. Children are made to experience the joy and wonder of learning.	4 3 2 1

2. Open classroom can be systematically organized and held within various environments. Number of observers and structure of the group of observers can vary. Lesson can be observed by just one person, usually master teacher. Group of observers can be consisted on school, regional or national basis and can include not only teachers but university lecturers and educational experts. If the group of observers is large, class can be placed in school gymnasium or lesson can be transmitted via school closed circuit television. Since classroom visits following the Lesson Study movement occurred in Japan on a regular basis from 1873, Japanese pupils have become accustomed to studying in such open setting (Sarkar Arani at all, 2010). Teachers who are observing the lesson examine the responses and behaviour of pupils to determine the degree of their interest in the lesson, and the suitability of the questions asked and of the resources used in teaching.

3. There are various themes of Lesson Study such as developing mathematical thinking, learning for/by themselves, development, reform or improvement of the curriculum. The chosen study topic depends on the various dimensions and focus of the open classroom and teachers' groups. The objective is specified at each class level in relation to the curriculum. The objective is often described by a sentence such as this: "Through A, students learn/ understand/are enabled to do B." Both learning how-to (A) and achievement (B) are objectives of curriculum. In the Lesson Study cycle, discussion and reflection are done after observation. It is necessary to talk about study topics and objectives, questioning the reason for each teacher's behavior in the teaching process.

4. The format of the lesson plan is not fixed but is usually developed or improved depending on the study topic. Lesson Study is implemented to generate new lesson plan formats and new teaching approaches avoiding uniformed forms of lesson plans. This means that the different lesson plan formats take into account the differences that arise from the local theories which are used to explain lesson planning and answer the questions about what, why and how we plan teaching and learning in a mathematics lesson.

5. Lesson study is conducted by teachers for developing pupils in a classroom and making each pupil developing him/herself. So, the focus is on pupils and not on researchers who just observe a classroom through their own lenses. In this sense, lesson study recommends that teachers are researchers who analyse pupils' understanding. At the same time, researchers are teachers who propose improvement (Isoda, 2011).

6. Usual consideration of Lesson Study is achievement in relation to study topic and objective. One of the most sharable products is description of model approaches. As results of Lesson Study we can find guidebooks for teaching contents and teaching approaches written by teachers. Also, videos have lately been used for sharing good approaches and practices by making them more visible. In the context of Lesson Study, a model approach means an illuminating approach and major resources for adapting a model into each teacher's classroom practice (Isoda, 2011).

7. The Lesson Study cycle continues beyond the present generation of teachers in the group. It is usually open to newcomers. Since Lesson Study is about teachers' daily activities, they repeatedly meet to discuss similar contents, themes and objectives of their lessons. However, during their teaching career the roles they take while participating in such projects differ. During one period a teacher may be a beginner but during an other, a facilitator in a group. Taking this into account, it is common over time to recognise similar experiences with the new challenges. That is the reason why Lesson Study brings the learning community beyond the present generation and why it is recognised as a reproductive practice in Japan, even if teachers are challenged with similar tasks and within different classroom settings.

8. Even though Lesson Study is recognised as a teacher activity, it is, particularly in the case of the Japanese elementary school, seen as an activity of pupils. The open class which provides a scene for Lesson Study is not reserved for educators but is open also to parents. Pupils follow and try to follow their teacher's activity in order to show their progress to others, including perhaps their parents. In this particular aspect, we can recognise that Lesson Study has an additional function, and that is to develop pupils' ability to learn by/for themselves and in full view of their careers.

The product of Lesson Study – Problem Solving Approach

As Japanese teachers cannot recognise the theory of Mathematics education without considering their practice, Japanese Lesson Study functions as teachers' research activities. In Japan new theories of teaching approaches and curriculum development are recognised as products of Lesson Study. There are many guidebooks that offer theoretic advice to teachers to (re)produce good lessons. However, the movement to develop the theory of mathematics education which support the development of the curriculum and the teaching/learning practice is not limited to Japan. Even though outside of Japan Lesson Study is often perceived as professional development process, there is a tendency to recognise Lesson Study as research activity that could produce new theories in teaching approaches or theories of curriculum development.

One of teaching approaches and consecutive theories that arises from Japanese Lesson Study is Problem Solving Approach known as the process through *posing a problem*, *independent solving*, *comparison and discussion*, and *summary and application*. This approach became known in the world through the TIMSS Video Study in the 1990s (Stigler, Hiebert, 1999). Becker and Shimada (1997) explained the approach from the perspective of open-ended problems. Shimada's idea itself originated in the 1940s.

The basic principle of the Problem Solving Approach is to nurture children's learning of Mathematics by/for themselves, that is to develop children's ability to think and learn Mathematics by/for themselves. The Problem Solving Approach is the method of teaching used to teach content such as mathematical concepts and skills, and mathematical process skills such as thinking, ideas, and values (Isoda, Katagiri, 2011). The Problem Solving Approach distinguishes a problem - a task given by teacher, and a problem posed by pupils. This approach is not supposed to teach how to solve a problem (a task) but teaches how to approach a problem solving activity. The goal of lesson is achieved through problem solving. (Inprasitha, 2006). Glenn at al (2000: p.20) explained the Japanese teaching approach as follows: "In Japan, closely supervised, collaborative work among students is the norm. Teachers begin by presenting students with a mathematics problem employing principles they have not yet learned. They then work alone or in small groups to devise a solution. After a few minutes, students are called on to present their answers; the whole classworks through the problems and solutions, uncovering the related mathematical concepts and reasoning." Table 3 shows a model of Problem Solving Approach through teaching phases.

Teaching phases do not imply teaching step by step, neither are they obligatory. They could be modified based on circumstances related to topic, classroom, aims and goals of lesson etc. Even though there are variations, the phases are fixed for explaining the ways to develop mathematical thinking in class (Isoda, Katagiri, 2011). As it is noticeable from Table 3, Japanese teachers play several roles at each stage of their lessons.

Table 3. Phases of the class for Problem Solving Approach

PHASE	TEACHER'S ROLE	PUPILS' STATUS
Reviewing the previous lesson	*Asking questions related to previous lessons*	Bring in focus on mathematical ideas and learned facts.
Posing the problem	*Posing the task with a hiden objective*	Given the task in the context but not necessary to know the objective of the class.
Planing and predicting the solution	*Guiding the pupils to recognise the objective; hatsumon.*	Having expectations, recognising known and unknown (problems) and approaches to problem solving. Recognise and identify objectives of the lesson.
Independent (group) solving/ executing solutions	*Supporting individual work - kikan-shido*	Bring in ideas to work on the task. In order to present some explanations, clarify and bridge the known and unknown in each approach. Try to present better ways.
Explanation and discussion, comparision and validation	*Guiding discussion based on the objective; hatsumon.*	Explaining each approach, compare approaches based on the objective of the lesson. Communicate in order to understand different ideas. Consider different ways in obtaining solutions and conclusions.
Summarization; application and further development	*Guiding the reflection - neriage*	Recognise and understand what students have learned. Appreciate their own achievement, ideas and ways of thinking. Re-evaluate contents through applying experience in new circumstances.

Teachers' key roles are described by special pedagogical terms such as *hatsumon*, *kikan-shido*, *neriage*, *bansho* and *matome*. These can be described as follows.

a. Asking a key question in order to provoke students' thinking at a particular point in a lesson is known as *hatsumon*. At the beginning of the lesson, the teacher may ask a question for probing or promoting students' understanding of the problem. In a whole-class discussion, on the other hand, teacher may ask about the connections in between the proposed approaches to the problem or the efficiency and applicability of each approach.

b. *Kikan-shido* means an instruction at students' desk, the one-to-one discussion. Teacher is purposefully scanning students' problem solving on their own in the way that he/she is moving about the classroom, monitoring students' activities mostly silently and doing two important activities that are closely tied to the whole-class discussion that will follow. First, teacher assesses the progress of students' problem solving and if necessary suggests a direction for students to follow or gives hints to the students for approaching the problem. Second, he or she will make a mental note of several students who made the expected approaches or other important approaches to the problem. They will be asked to present their solutions later (Shimizu, 2006).

c. The term *neriage* acts as a metaphor for the process of "polishing" students' ideas and getting an integrated mathematical idea through a whole-class discussion. Based on his/her observations during *kikan-shido*, the teacher carefully- in particular order, calls on students, asking them to present their methods of solving the problem on the chalkboard. The order is quite important to the teacher both for encouraging those students who found naive methods and for showing students' ideas in relation to the mathematical connections that will be discussed later (Shimizu, 2006). An incorrect method or error may be presented in cases when teacher figures out that it would be beneficial for the class. Students' ideas are presented on the chalkboard, to be compared with each other with oral explanations. Teacher is supposed to guide the discussion by the students towards an integrated idea and to avoid pointing out the best solution. Japanese teachers regard *neriage* as critical for the success or failure of the entire lessons (Shimizu, 2006).

d. Effective use of blackboard is adressed as *bansho*. In Japan, the blackboard is used extensively in lessons: to keep a record of the lesson, to help students remember what they need to do and to think about, to help students see the connection between different parts of the lesson and the progression of the lesson, to compare, contrast, and discuss ideas that students present, to help to organize student thinking and discovery of new ideas (Takahashi, 2006) teachers use the blackboards. Teachers usually try to keep all that is written during the lesson on the blackboard without erasing it if possible, which gives both the students and teacher a birds-eye view of what has happened in the class at the end of each lesson. From the learner's perspective, it is easier to compare multiple solution methods if they appear on the blackboard simultaneously.

e. *Matome* means summing up, i.e. reviewing briefly what students have discussed in the whole-class discussion and summarising it by the teacher.

Conclusion

Significance of the Lesson Study goes far beyond professional development of teachers, as it is usually perceived outside Japan. Beside individual professional development, research lessons contribute to spread new content and approaches, connect individual teachers' practices to the school goals and broader goals, create demand for improvement, shape national policy and teach teachers to understand children better.

Recently, educators in many countries have begun to learn from their Japanese counterparts how to develop a new culture for promoting learning communities at their schools (Sarkar Arani et al., 2009). Collaborative lesson plans, participant observation, and reflective thinking on teaching are the three points that serve as the core of Lesson Study and should be highlighted in its application. It is essential that educators, as equal participants, clarify their own views toward education and pupils' learning. They have to present a collaborative lesson plans based on these aspects, and articulate their fundamental approaches to teaching. The final consideration is that Lesson Study should be understood as both regular practice and as a process, and that problems will not be resolved after a single session. Effective lesson study follows the teaching of pupils and their progress over a long period of time.

In the case of teaching and learning mathematics Lesson Study can shed a completely new light on how Mathematics is learnt by children and how the learning experience can be organized to help children not only to carry out techniques but make sense of the mathematics and focus on what is important to make mathematics both powerful and simple at the same time (Tall, 2008). Lack of understanding at one stage usually makes successive stages more difficult, leading to rote learning. Also, too much practice too soon can be ineffective or lead to math anxiety (Isaacs, Carroll, 1999). Lesson Study involves the careful design of good lesson sequences that focus on helping children develop different methods of approach from which they can find ways of working that are fast, easy and accurate. The lesson spends a significant amount of time reflecting on what makes sense and focusing on finding better ways of working. For the teacher, it requires not only knowledge of Mathematics, but also deep experience of how children think as they learn Mathematics (Tall, 2008). Lesson Study is not primarily focused on teaching techniques, but on upgrading conceptual understanding to level which lead to nourishing pupils who learn Mathematics by/for themselves. This requires an understanding not only of the Mathematics itself but how it can be organised to make sense in the long-term process of learning the subject.

Lesson Study in Mathematics goes beyond the design of individual lessons to the development of a long-term teaching approaches which are dedicated to understanding the nature of mathematical thinking. Lesson Study in Mathematics is fundamentally purposed to improve the nature of students' mathematical thinking, including not only the ability to perform routine tasks accurately and efficiently, but also to develop the abilities and attitudes to solve novel problems by thinking mathematically in new situations. As the result, new teaching theories have arisen, such as The Problem Solving Approach. In this way, Lesson Study is recognised as a reproductive science for teaching in the classrooms (Isoda, 2011). Therefore, there is a continued need for further implementation and future research on the lesson study model and its products. Recently, Lesson Study has become a widely-used and highly-refined methodology, not only in Japan, but also around the world.[3] As it moves into different countries, local versions of lesson study may develop, using the overall structure to its best effect in different locations. In the case of Balkan countries, Lesson Study may be initiated on the basis of a well-established practice of exemplary lessons.

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3 In US, the Lesson Study Research Group (LSRG) maintains a central database of U.S. lesson study groups, initiatives, and related activity. http://www.tc.columbia.edu/lessonstudy/. In UK, see http://lessonstudy.co.uk/the-lesson-study-process/. In Malaysia, the first Lesson Study project was initiated in 2004 (Lim, White, Chiew, 2005)

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др Кармелита Пјанић
Педагошки факултет, Универзитет у Бихаћу, Босна и Херцеговина

Порекло и производ јапанске „студије часа“

Циљ овог рада је да се прикаже „студија часа“ и њен главни производ – „приступ решавања проблема“, заснован на релевантном проучавању литературе и директној опсервацији аутора рада. Јапанска „студија часа“ се препознаје као успешна методологија у математичком образовању. У западним земљама „студија часа“ се обично схватала као професионални развојни процес који је укључивао јапанске наставнике да систематично испитују своју праксу ради њене веће ефикасности. Међутим, „студија часа“ је више од професионалног развоја. Реч је о наставној активности заснованој у науци и у методологији осамдесетих година 19. века (Isoda, 2011). Порекло „студије часа“ може да буде забележено у образовној пракси у периоду Меији у Јапану, где су људи посматрали начине поучавања да би знали како да воде процес поучавања и учења. Од тада „студија часа“ функционише као средство које омогућава наставницима да развијају и проучавају властиту наставну праксу и да учине познатијим локалне теорије образовања. Овај развој се одвија због димензија које карактеришу ’студију часа’ као процес колаборативних активности и истраживања које није примарно фокусирано на технологију и средства, већ на садржај и ученике. Производи „студије часа“ нису само ограничени на оно што су учесници научили из одређене рефлексивне дискусије у оквиру часа и после њега. То такође укључује развој локалних теорија у математичком образовању. „Студија часа“ функционише као репродуктивна наука која нам приближава локалне теорије у математичком образовању, које су препознате са наставном праксом која произлази из њих. Једна од теорија поучавања математике, која се појавила из „студије часа“, јесте „приступ решавања проблема“, познат као јапански начин поучавања и теорија поучавања и учења (Stigler & Hiebert, 1999). ’Приступ решавања проблема’ је приступ поучавања који се користи да би се формирали математички појмови и развиле математичке вештине – математичко мишљење, идеје и вредности (Isoda & Katagiri, 2011). Основни принцип „приступа решавања проблема“ је да се деца науче да уче математику сама, тј. да се настоји у томе да се развијају деца која размишљају и која уче математику за себе. Однедавно је „студија часа“ постала широко примењена не само у Јапану већ и широм света. Пошто је све више присутна у разним земљама, локалне верзије „студије часа“ могу да се развијају коришћењем комплетне структуре на најбољи начин на разним локацијама. У случају балканских земаља, „студија часа“ може да буде базирана на добро заснованој пракси огледних часова.

Кључне речи: „студија часа“, „приступ решавања проблема“, математичко образовање.

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Iordanka G. Gortcheva[1], PhD
Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences, Department of Education, Bulgarian Academy of Sciences, Sofia, Bulgaria

Mathematical and cultural messages from the period between the two world wars: Elin Pelin's story problems

Abstract: *In the mid-thirties of XX c. the renowned Bulgarian writer Elin Pelin published the children's newspaper Path (in Bulgarian: Пъ̄ѿека) regularly including mathematical story problems. Their unusual imagery and plots were quite different from textbook math problems. The leading character Old-hand Stanyo loved to tell his neighbours' children stories in which he intertwined elements of mathematics or logic. Too lengthy sometimes, the story problems revealed various sides of culture to children and taught them to pay attention to detail, think logically, and not to neglect common sense. Whatever life situations were described, the moral was always the same – the knowledge of mathematics helped overcome obstacles. Elin Pelin's newspaper was circulated all around Bulgaria inspiring children from towns big and small to learn counting, divisibility of numbers, systems of linear equations, binary number system, etc. These topics continue to be in the scope of interests of contemporary students, teachers, and mathematics educators. Story problems of Elin Pelin's series, exemplary pieces of which are analyzed in the article, attract modern readers with the unique mathematical and cultural values of their time.*

Key words: *Elin Pelin, primary school, story problems, culture.*

Introduction

Mathematics curriculum for Bulgarian primary schools is based on the decimal number system. The pupils perform elementary arithmetic operations while solving numerical examples, simple linear equations or inequalities, and story (or word) problems. In mathematical story problems, numerical data and logical relations are interwoven in real life situations described through a narrative. To deal with them successfully, the primary school children need to acquire comprehensive reading skills. They help them separate what is given from what is sought, recognize mathematical concepts and decide what operations to perform. In essence, these activities are mathematical modeling, which makes solving story problems crucial for building the little students' modern mathematical literacy.

1 gorcheva@math.bas.bg

For years the researchers in mathematics education like Pollak (Pollak, 2007), Niss (Niss, 2012), Stillman et al. (Stillman et al., 2013), Spandaw and Zwaneveld (Spandaw and Zwaneveld, 2010) and many others have been emphasizing the importance of teaching and learning of mathematical modeling in school. The authors of mathematical textbooks make significant efforts to prepare the pupils for the challenges of problem solving: after each unit they offer numerous story problems, which differ by the level of difficulty and allow the teachers to meet their students' needs. Math competitions have had a significant share in raising the students' mathematical literacy as well. At the beginning of each school year the Bulgarian Ministry of Education and Sciences issues an official schedule of all upcoming competitions and olympiads.[2] To train their students, the mathematics teachers organize circles where the children not only solve problems and learn extracurricular mathematical topics, but also make new acquaintances. In such a way mathematics turns into a subject that creates social networks among teachers, students, and parents.

The socializing effect of mathematical modeling for elementary schoolchildren was noticed and reported by Pollak (Pollak, 2007). Long before him, the Bulgarian man of letters Dimitar Stoyanov known by his pseudonym Elin Pelin also realized that very effect of mathematical problem solving. Although with no professional training in mathematics, he turned the spreading of mathematical literacy through a children's newspaper into his mission.

Writer Elin Pelin and his mathematical story-problems

According to Rothschild, Elin Pelin is Bulgaria's "most popular interwar author", who wrote "superb children's stories and poems" (Rothschild, 1974: 395). Elin Pelin (in Bulgarian: Елин Пелин, 1877-1949) dreamed to be an artist, but became one of the most renowned Bulgarian writers who painted his images in words. Since the age of 17, for several years he worked as a teacher in his native village where he watched the children play, study and interact with their peers. This first hand experience inspired many of the fascinating plots and characters in his later stories.

Elin Pelin was also a publisher. In the mid-1930s he issued the children's newspaper *Path* (in Bulgarian: Пътека) where he regularly included mathematical story problems. They are now collected in a small book (Elin Pelin and Podvarzachov, 1992) published posthumously by his son. The main character in these stories is Old-hand Stanyo who asks his neighbors' children to solve problems that test their logical thinking, wit, and attention to detail. They often exceed in length and totally differ from ordinary textbook problems. Most of them are fairly easy to solve; others challenge even professional mathematicians and educators. What unites them is their reverence to mathematical literacy, common sense and nuances in speech. The stories are evidence of the type of mathematical knowledge the common people of the time regarded helpful to overcome the obstacles of life. I implemented many of them in my seminars with the prospective primary school teachers from the Department of Preschool and Primary school education at Sofia University "St. Kliment Ohridski".

Elin Pelin's approach to problem formulation

All students I worked with knew a lot about Elin Pelin's literary works, but no one had heard about his story problems. The example I used as a warm-up activity was more of a story than a mathematical problem and vividly demonstrated the culture of farmers' life and the nature of everyday problems at the time:

2 Bulgarian Ministry of Education and Sciences. Order No. 1505 from September 30, 2014 about School olympiads and national competitions. Retrieved October 10, 2014. from http://www.mon.bg

Story problem 1. "It was last summer," Old-hand Stanyo began his story. "I was walking down the Pirin mountain, heading for the plain. On a paddock I saw two horses grazing. The stud was tall and black, robust, with a broad saddle and sleek fur. Although his tail was missing, he looked like a pedigree animal. The mare was very beautiful too – of a local breed, a little short, immaculately white, you just could not help stroking her, with streamers intertwined to decorate her magnificent tail. A really gorgeous animal!

A few steps away, three guys were bickering about something, waving their arms… It turned out they were having an argument. Two of them were travelers. They had come to that place at noon in the sweltering heat and left their horses to graze on the paddock while they took a nap in the shadow. Meanwhile the owner of the paddock came from the nearby village, woke them up and demanded that they pay him for the grazing of their horses. The travelers agreed. One of them dug into his pocket and produced 10 levs[3] which the villager willingly took. Yet when the other traveler handed him his 10 levs, he refused to accept it."

"It is too little," he said. "Ten levs for the stud is a fair deal, but not for the mare. For her, if not 20, you should pay me at least 15 levs."

No matter how hard they tried to persuade him, he would not agree. Therefore they summoned me to settle the dispute.

"Well, and did you, uncle Stanyo?"

"I said straight away that the villager was right beyond a shadow of a doubt. Why do you, buddies, think I supported his opinion?" (Problem 4, Elin Pelin and Podvarzachov, 1992: 15-16)

This small literary masterpiece not only excited the future primary school teachers, but also puzzled them: they were not sure whether it can be classified as a mathematical story problem. The following speculations by Novakova (Novakova, 2003) shed light on the case, allowing the students to form their own opinions: "In order to have a story problem, there has to be a requirement (a question) that defines what is sought." (Novakova, 2003: 165). In Story problem 1 such a question existed. The answer was also provided, which was not quite usual for problem formulation. Elin Pelin did it for a purpose – to boost the logical thinking of his readers. My audience reasoned out loud that the mare had to eat up more grass than the stud since more money had been demanded for her grazing. One of the students noted that in the wild animals do not overeat. Thus the idea that the mare could have been pregnant came up and the prospective teachers, most of whom were female, adopted that explanation. Their inference was plausible and Old-hand Stanyo would have probably accepted it. However, his reasoning differed: "The stud's tail was missing while the mare's tail was long and even had streamers for decoration. Thus the mare was able to easily keep the flies away and graze undisturbed, while the stud, continuously being pestered by the insects was able to pasture less. Therefore, the villager was right to want more money for the mare's grazing." (Elin Pelin and Podvarzachov, 1992: 80).

The students did not expect such a turn of events. They became aware how skillfully Elin Pelin had led them to this conclusion: describing the stud's *missing* and mare's *magnificent* tail, he was giving them a key to his point of view. They realized that writing his story and formulating the problem, the great writer had in mind both the concept and the solution. Thus the future teachers received a lesson in paying attention to detail and context.

One of Elin Pelin's noble ambitions as a publisher was to bring the culture of the world to the poor children from the smallest villages in Bulgaria. He created a story in which he told them about the enchanting nature of the Far East. But the readers would soon realize that the landscape and animals were just a background:

3 Lev (in Bulgarian: лев) is the name of Bulgarian currency.

Story problem 2. Once upon a time in China, on the bank of the Great River Yangtze, a boatman named Ha-de lived with his 10-year-old son Pu-pi. Ha-de built himself a large and strong boat and made his living by carrying bricks to the opposite bank of the river. There the mandarin Ah Ti-Vrag was having a new palace built.

The mandarin was very mean. One day he called Ha-de and told him:

"Listen to my order! Tomorrow you must tell me how much my favorite elephant Sambo weighs. Otherwise you'll be in trouble."

The boatman panicked: he did not have such big scales to weigh an elephant on them. Desperate, Ha-de told his son about this impossible task. Pu-pi just laughed; he was sure that his father could deal with it. And here is what happened: the next day Ha-de stepped before the mandarin and told him exactly how much his favorite elephant Sambo weighed.

How do you think the boatman and his son managed to find the weight of the elephant? (Problem 1, Elin Pelin and Podvarzachov, 1992: 8-9)

The situation described was again unexpected for the undergraduate students and for a while they pondered how to approach the problem. Another reading of the text and the practice with the previous problem led them to carefully interpret the information which the expressions "he built a large and string boat" and "he did not have such big scales" carried. They concluded that the boatman had to have some scales, although not big enough to weigh an elephant on them, but pretty good to identify the weight of a brick, for example. Thus they almost reached the solution Elin Pelin had in mind: "Pu-pi, the boatman's son, hauled the boat into the water, lured Sambo inside it and drew a line on the boat's outer side to mark the level of the water. Then Pu-pi took Sambo back to the bank and began to load the boat with bricks until the water reached the mark; then he weighed one brick on the scales, counted the bricks in the boat and calculated the elephant's weight." (Elin Pelin and Podvarzachov, 1992: 80).

I asked the students to collectively formulate Elin Pelin's problem as it could be written for a modern primary school textbook, and make a comparison with the original version. The text unanimously suggested was:

Students' version of Story problem 2. A single brick weighs 3 kg. How many kilograms do 1,200 such bricks weigh?

The future teachers' comments on their own version were that it was "mathematically precise, doable in a minute, free of emotion and boring", because "no insight is needed for this problem to be solved." One student expressed an interesting opinion: "Through this story problem Elin Pelin sends a message to the children that mathematics is not an isolated subject. That is what is missing in our version."

The literary works which Elin Pelin left document various aspects of common people's life in the period between the two world wars, including their financial hardships. As the text of Story problem 1 reveals, they count every penny and negotiate each bargain. Understanding the importance of interpersonal relations which occur at the markets, the writer uses this environment as a setting for several of his story problems. From Old-hand Stanyo's next story contemporary readers learn that buying chocolate to a child was a rare event: a reward or maybe a bribe for proper behavior. The humorous plot aims not only to provide mathematical knowledge to the children, but to instill good manners in them and teach them respect for parents and grandparents:

Story problem 3. My grandson Mitko is a very spoilt child. This morning my wife and I headed for the market. As loving grandparents do, we took Mitko with us to spend the day together, but he became a pain in the neck: whichever store we stopped by, Mitko always wanted us to buy him a present.

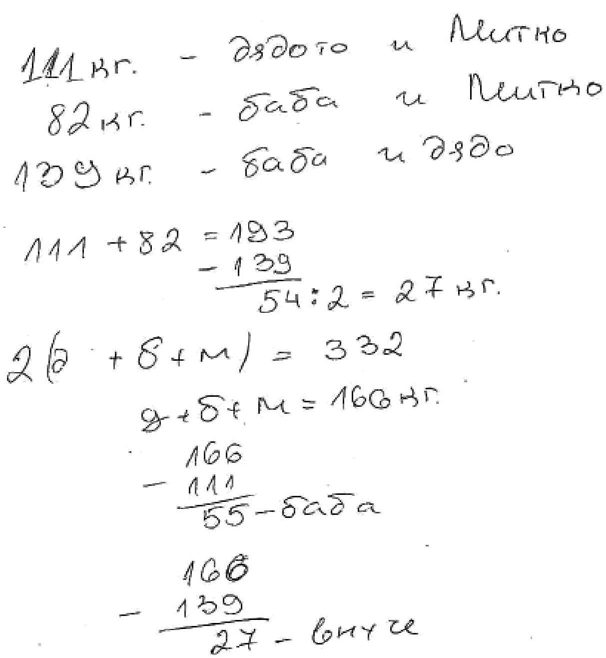
In a grocery store I accidentally saw big scales and asked the seller to check my weight. I was just about to step on the scales, when Mitko grabbed my pants:

"Grandpa, I want to be with you on the scales!"

"Let me first weigh myself, buddy. Then I'll weigh you as well."

No matter how hard I tried to persuade my grandson, even by buying him chocolate – he did not agree to get off the scales. I was forced to give in. I allowed Mitko to weigh with me and the scales showed 111 kg. After we got off them, I called my wife to quickly weigh herself while I was entertaining the troublemaker Mitko. But he caught her skirt and did not drop it until he stepped on the scales again with her. Thus both Mitko and grandma's weight together came exactly to 82 kg.

"So you and your wife did not weigh separately, uncle Stanyo?"



111 kg – grandpa and Mitko

82 kg – grandma and Mitko

139 kg – grandma and grandpa

$$\begin{array}{r} 111 + 82 = 193 \\ -\underline{139} \\ 54 \div 2 = 27 \text{ kg} \end{array}$$

2(grandpa + grandma + Mitko) = 332

grandpa + grandma + Mitko = 166 kg

$$\begin{array}{r} 166 \\ -\underline{111} \\ 55 \end{array} \text{ – grandma}$$

$$\begin{array}{r} 166 \\ -\underline{139} \\ 27 \end{array} \text{ – grandson}$$

Figure 1. A solution to Story problem 3 by a prospective primary school teacher (left – the original excerpt; right – the English translation)

"No, we didn't. We barely managed to get together on the scales. Thus both grandma and grandpa, without Mitko, weighed 139 kg. Tell me now, how many kilograms each and every one of the three of us weighs." (Problem 16, Elin Pelin and Podvarzachov, 1992: 45-46)

Probably the most straightforward way to approach this problem is to use a system of linear equations, but the topic is not included in Bulgarian primary school mathematics curriculum. Therefore the undergraduate students were to step into children's shoes.

As Figure 1 shows, the solution process was not smooth for everybody. Pretending to have only a little pupil's mathematical knowledge was tense for the particular prospective teacher: she obtained the same answer twice (Mitko's weight), but unintentionally omitted to point out the third answer required (the grandfather's weight).

Elin Pelin loved to be in line with the latest developments in science and technology. In an unpublished telephone interview from 2006 with Elin Pelin's son – Mr. Boyan Dimitrov, – I learned a lot about the great writer's interest in science and technology. He subscribed to the journal "Science" and took many of his ideas from it; he assembled three radio sets all by himself; he regularly went to the market and as he was a keen observer and an attentive listener, he probably borrowed ideas and characters to later implement in his story problems. In the last narrative discussed here the knowledgeable reader will recognize the characters from the fairy tale "One-eye, Two-eyes, and Three-eyes" published in 1812 by Brothers Jacob and Wilhelm Grimm (Grimm, 1909-1914). What makes Elin Pelin's story unique is its mathematical content that cannot be found in the original Brother Grimm's tale.

Story problem 4. Once upon a time there lived an old and cruel witch. She had three daughters. The first had only one eye placed in the middle of her forehead; the second had three eyes and the third two eyes, just like other people do. The old witch loved the One-Eyed and the Three-Eyed daughters, but hated the Two-Eyed one: maybe because she was the most beautiful and her two eyes were just in their right places.

One day the witch called her three girls and gave them apples: 50 wonderful apples to the One-Eyed, 30 wonderful apples to the Three-Eyed and only 10 unripe and wrinkled apples to the Two-Eyed.

"Now, girls, go immediately to the market and sell all your apples," she ordered them. "However, make sure that each of your brings back the same amount of money!"

The Two-eyed daughter was scared: "O Mom! I cannot earn as much money for my wrinkled 10 apples as my sisters for their pretty 50 and 30 apples."

"Shut up!" chided the old hag. "If you bring even a penny less than your sisters, you will be in trouble!"

The Two-Eyed girl began to cry: "Then let me sell my apples at a higher price than them."

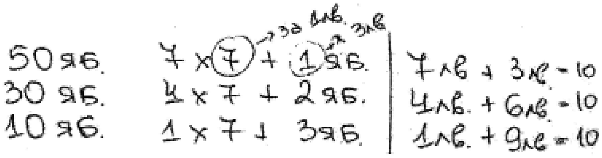
The old woman scolded again: "Listen! If you dare sell at a price even a penny higher than your sisters' prices, I will punish you. Go!"

The three girls headed for the market: The One-Eyed and the Three-Eyed sisters, dressed up and dolled up, ran ahead, laughing. Their Two-Eyed sister, wearing her only faded cotton dress, dragged behind, crying miserably. Thus the wretch was left a whole kilometer behind her sisters. Meanwhile, her beloved golden-horned goat caught up with her and turned the ten wrinkled apples into ten irresistibly fresh and juicy ones.

What happened later? Soon the Two-Eyed sister reached the market and stood there next to her sisters. Whatever price they had asked for their apples, the same price she wanted for hers. At the end of the day, although The One-Eyed sister sold 50 apples, The Three-Eyed sold 30 apples, and The Two-Eyed one sold only 10 apples, the three girls earned an equal amount of money and brought it back to their mother.

At what price, do you think, the three sisters sold their apples to fulfil the demands of their witch mother? (Problem 9, Elin Pelin and Podvarzachov, 1992: 29-30)

When the prospective primary school teachers read the problem from their handouts, they were puzzled saying it was unsolvable. Several female students suggested reading it again, in parts, changing their voices appropriately. The careful analysis of the text revealed that a clue to model the situation mathematically still existed: it was to decipher not only what was told to the readers, but also to guess what was not told. The only thing the mother required was that all the sisters demanded equal prices for their apples at the market. For example, they were not forbidden to change the prices. Thus a two-stage sale of a different number of apples could bring the same revenue to the girls.



per 1 lev 3 levs

50 apples	7 x 7 + 1 apple	7 levs + 3 levs = 10
30 apples	4 x 7 + 2 apples	4 levs + 6 levs = 10
10 apples	1 x 7 + 3 apples	1 lev + 9 levs = 10

Figure 2. Elin Pelin's solution to Story problem 4, written by a student (left – the original excerpt; right – the English translation)

Elin Pelin provided such a solution which I showed to the audience. We discussed it for a long time because the undergraduate students had difficulty believing it was possible. In the end they realized the problem had a solution (Figure 2).

The students' surprise was even bigger when I demonstrated a few more solutions obtained by the method of linear programming (Kelevedjiev and Gortcheva, 2010). The students were fascinated by the existence, as well as by the potential of this method. Tables 1 and 2 show two such solutions.

I assigned the undergraduate students to create again their formulation of the problem. Collectively, they produced the following text:

Students' version of Story problem 4. A retailer rents a store in three city locations and pays the same rent for each. Today he sells top fashion T-shirts, at the same prices in all three stores. In the first store there are 50 T-shirts to sell, in the second – 30, in the third – 10. As the rents he pays are equal, he wants to have the same revenue from each store, keeping the prices equal at every moment of the sale. At what prices can the retailer sell the T-shirts in order to ensure the same revenue from the three stores?

Table 1. A sale in which every sister can earn 70 levs

Daughter	Number of apples	First price	Apples sold	Second price	Apples sold	Revenue
One-eyed	50	1 apple per 1 lev	48	1 apple per 11 levs	2	48 levs + 22 levs
Three-eyed	30		26		4	26 levs + 44 levs
Two-eyed	10		4		6	4 levs + 66 levs

Table 2. A sale in which every sister can earn 100 levs

Daughter	Number of apples	First price	Apples sold	Second price	Apples sold	Revenue
One-eyed	50	1 apple per 1 lev	45	1 apple per 11 levs	5	45 levs + 55 levs
Three-eyed	30		23		7	23 levs + 77 levs
Two-eyed	10		1		99	1 lev + 99 levs

Asked to compare the two versions of the problem, the undergraduate students, mostly young ladies with vast shopping experience, found the T-shirt version more realistic. They eagerly discussed: "Even if we hadn't been able to reach such exact solutions, we would have invented at least a strategy to them."

Discussion of the results

The selection of Elin Pelin's story problems which I present herein gives an idea about the writer's interest in mathematics and sciences, but their scope remains uncovered. Fortunately, I managed to introduce the majority of the story problems to the students at my seminars. They all enjoyed his ingenuous ways to imbed concepts like divisibility of numbers, binary number representation, Diophantine equations, graph theory, permutations, or Cartesian coordinate system in fabulous stories about animals, rascals, buckets of water, navy ships… Although not so frequently, ideas from science and technology were also found there, as well as literature related tasks.

Initially I was concerned whether using such old-fashioned story-problems was justifiable in 21st century teacher education and in its projection – primary school education. Confirmation came from the students. Here is how they reflected on them:

*Student 1***:** "I never imagined that Elin Pelin could write math problems for children."

*Student 2***:** "I changed my opinion about men of letters which only at first glance did not understand maths."

*Student 3***:** "From a literary point of view the writer did a perfect job: while Old-hand Stanyo talked, the children listened to him to realize in the end that it was not just a story, but a problem to be solved. As a reader I felt the same way."

*Student 4***:** "If all word problems had been represented in such manner, mathematics would have been my favourite subject at school."

The fact that a group of undergraduate students started impulsively reading out loud Story problem 4 because it seemed unsolvable to them can be interpreted as a form of "expressive engagement" (Sipe, 2002): by immersing themselves in the plot the prospective teachers tried to "dominate the text" (Sipe, 2002), i.e. to solve the problem.

I observed the same motivating effect on the students of all Elin Pelin's story problems. In my opinion, to great extent it was due to their unique cultural messages which my audience with a background primarily in humanities highly valued.

As reported by Lindgren and Bleicher (2005), many future elementary school teachers find science content courses "boring or difficult, with little relevance to their lives or to teaching children" (Lindgren and Bleicher, 2005). Therefore, situations like the one described in Story problem 2 boost students' interest in science and make them search for more practical problems and connections among school subjects.

My experience shows that Elin Pelin's story problems can be successfully used even as a source for a drama project in primary school (Gortcheva, 2014). For such a purpose, it is not necessary to literally use the texts, but only the way mathematical ideas have been intertwined.

Concluding remarks

Through their specific contents Elin Pelin's story problems help contemporary readers touch the culture and lifestyle of common Bulgarian people from the period between the two world wars and learn about mathematical knowledge and skills they needed most. Properly balanced, these problems still have place in modern primary school education, as well as in core subject courses for prospective teachers. They challenge even professional mathematicians to model the nonstandard situations described. The multiple solutions which some problems allow make them appropriate for implementing inquiry based learning in the classrooms.

Elin Pelin's story problems can serve teachers to effectively combine the traditional "3Rs" of primary school education: Reading, wRiting and aRithmetic. The reading skills developed by such texts are crucial for children's success in problem solving. By focusing both on the context and detail, they help pupils analyze situations, find numerical and logical data in texts, get rid of unnecessary information, and answer questions. To communicate their mathematical knowledge, students should possess strong writing and oral skills. Only then will they be able to persuasively express their opinions and results on paper, in files, presentations, blogs, social networks, podcasts... And here come to help the messages of Elin Pelin's story problems to their young readers: to study hard in order to have things to say and write about.

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др Јорданка Гочева

Институт за математику и информатику бугарске Академије наука,

Одељење за образовање, Софија, Бугарска

Математичке и културне поруке из периода између два светска рата – проблеми текстуалних задатака Елина Пелина

Математички курикулум бугарских основних школа је заснован на децималном бројном систему. Ученици обављају основне аритметичке операције са бројевима, решавају једноставне линеарне једначине и неједначине, као и текстуалне задатке. Код ових задатака преплићу се бројчани подаци и логички односи у свакодневним ситуацијама, што је описано у раду. Да би успешно решавали текстуалне задатке, ученици морају да направе разлику између онога што је дато и онога шта се тражи, да препознају математички појам у тексту и да одлуче које математичке операције треба да користе да би дошли до решења. С обзиром на то да је основа ових активности математичко моделовање, решавање текстуалних задатака је основно за формирање математичке писмености ученика нижих разреда. Годинама су истраживачи математичког образовања, као што су Полак (Pollak, 2007), Нис (Niss, 2012), Стилман и др. (Stillman et al., 2013), Спандау и Званевелд (Spandaw and Zwaneveld, 2010) и многи други, истицали важност поучавања и учења математичког моделовања у школи. Аутори модерних математичких уџбеника су начинили велике напоре да припреме ученике за изазове решавања математичких задатака – после сваке лекције понудили су много текстуалних задатака који се разликују по нивоу и тежини и тиме омогућавају наставницима да помогну ученицима. И математичка такмичења су имала важан удео у побољшавању математичке писмености ученика. Да би спремили ученике за учешће на такмичењима, наставници математике су организовали секције у којима ученици не само да су решавали проблеме и учили о темама ван наставног програма већ су склапала и нова познанства. Тако се математика претворила у предмет који је омогућио дружење међу наставницима, ученицима и родитељима. Ефекат социјализације математичког моделовања код деце основношколског узраста приметио је и објавио Полак (Pollak, 2007). Давно пре њега, бугарски учењак без професионалног знања математике схватио је тај ефекат математичког решавања проблема. Тај интелектуалац је био Елин Пелин (1877–1949). Желео је да постане уметник, али је судбина хтела да слика речима. Према Ротшилду (Rothschild, 1974), био је бугарски „најпопуларнији аутор између два рата", који је писао „дивне дечје приче и песме“ (Rothschild, 1974: 395). Елин Пелин је прихватио ширење математичке писмености међу бугарским ученицима као мисију. Средином тридесетих година 20. века покренуо је дечји часопис „Стаза“ (на бугарском „Пътека”) (1932–1936), редовно објављујући математичке текстуалне задатке. Они су сада прикупљени у књижици (Pelin & Podvarzachov, 1992), коју је постхумно објавио његов син. Њихово сликовито излагање и суштина су били много различити од текстуалних задатка у уџбеницима математике. Главна личност, стара Шака Стањо, волела је да прича комшијској деци приче у којима су се преплитали елементи математике и логике. Понекад су били сувише дуги, али су текстуални задаци откривали различите стране културе деци и учили су их како да обрате пажњу на детаље, да логички размишљају и да не запостављају здрав разум. Какве год ситуације да су биле описане, поука је била увек иста – математичка писменост је помагала да се превазиђу животне препреке. Часопис „Стаза“ се читао по целој Бугарској и инспирисао је децу из великих и малих градова да уче да броје, да сазнају о дељивости бројева, системима и линеарним једначина, бинарном бројном систему итд. Стога су текстуални задаци Елина Пелина доказ врсте математичког знања обичних људи, које може да се сматра драгоценим. Мој рад са студентима на основним студијама и са постдипломцима показао је да, иако нису били модерни, текстуални задаци Елина Пелина су привлачили пажњу читалаца. У знатној мери, разлог томе су елементи културе у периоду између два рата и математичке поруке које њихове јединствене формулације и решења садрже. Као велики писац, Елин Пелин је предвидео шта ће бити занимљиво за следеће генерације и како ће деца гајити вредности као њихови родитељи, баке и деке.

Кључне речи: Елин Пелин, основна школа, текстуални задаци, математички појмови, култура.

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Aleksandar M. Nikolić[1], PhD
Faculty of Technical Sciences, University of Novi Sad, Novi Sad, Serbia

Original Paper

The work of Judita Cofman on Didactics of Mathematics[2]

Abstract: *Judita Cofman was the first generation student of mathematics and physics at Faculty of Philosophy in Novi Sad, Serbia, and the first holder of doctoral degree in mathematics sciences at University of Novi Sad. Her PhD thesis as well as her scientific works until the end of 70's belongs to the field of finite projective and affine planes and the papers within this topic were published in prestigious international mathematical journals. The aim of this paper is to draw attention to Cofman's contribution in didactics and teaching of mathematics through the activities with young mathematicians, to whom she devoted the second part of her life and scientific work. Her reflections on importance of geometry based on her experiences with high school students are specially pointed out.*

Key words: *Judita Cofman, teaching of mathematics, didactics of mathematics.*

Introduction

Mathematician Judita Cofman[3] was born in Vršac on 4th June 1936. She came from a well-known and formerly wealthy family of Zoffmanns whose arrival in Vršac is put at the time of the reign of Maria Theresa of Austria (1717-1780) where they came from a German region with a strong beer brewing tradition (Kuručev, 2007: 157-163). Although the Zoffmanns were originally German, they gradually adopted Hungarian identity, so Judita declared herself as a Hungarian from Vojvodina. An environment of material and cultural wealth marked the life of Judita's father Ákos Zoffmann (1910-1974). Having received wide education in Germany, he became a great expert in beer brewing, and wine growing and storing industries. Judita's mother Lujza (1910-2000), born Kozics, comes from a Hungarian family of lawyers from her father's side, while her mother came from Vršac. Lujza's grandfather was a mathematics teacher at the Vršac Grammar School, and her uncle was the mayor of the town of Vršac. Despite the incertitude and horrors of World War II, Judita enjoyed a happy childhood at her family home. She went to primary school in Hungarian and later to Serbian Grammar School in her hometown. The

1 nikaca@uns.ac.rs

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3 Her name was entered into the Official Register as Judit Zoffmann, but in the documents of Yugoslavia of that time her name was written in its Serbian rendition as Judita Cofman, and that was the name she used for the rest of her life.

family home, full of love and harmony, installed in her a great feeling that work, study, reading, as well as the knowledge of foreign languages are necessary preconditions for success in life. Besides being gifted for mathematics, Judita had a talent for languages, so, besides her mother tongue of Hungarian and the official Serbian language, as a child she learned German, Russian and, which was rare at that time, English. She later learned French and Italian.

Judita Cofman's PhD thesis as well as her scientific work till the end of 70's belongs to the theory of finite projective planes, Möbius planes and Sperner's spaces - a very up-to-date and lively mathematical field closely related to algebra and group theory. Her results within these topics were published in prestigious international mathematics journals[4] and were presented at high ranking conferences.[5] Her results complemented the results of many great geometricians of the early 20th century on the one hand, and on the other, the active follow-up and advancement of certain subfields of projective geometry rest upon her results (Nikolić, 2012). In the second period of 20 years, from 1980 till 2001, Judita was completely devoted to the mathematical education and didactics of mathematics and improvements in teaching process, especially working with gifted teenage mathematicians.

Studies and career

Judita Cofman started mathematical studies in 1954, graduating with the highest grades in 1958. She was among 66 students enrolled as the first generation of students of mathematics at the Faculty of Philosophy in Novi Sad. They were all studying to be teachers of mathematics. At that time the majority of classes were given by professors from the University of Belgrade, academicians Miloš Radojčić, Anton Bilimović, Radivoje Kašanin and Jovan Karamata (professor in Geneva at the time). Among the mathematicians from Novi Sad, there were Mirko Stojaković and Bogoljub Stanković, while the first assistants were Vojislav Marić and Mileva Prvanović. They were all to become eminent Serbian scientists and members of Serbian Academy of Sciences and Arts. The first elected Assistant Professor at the Mathematics Department was Mileva Prvanović (1956), who taught in the field of geometry. Judita Cofman was appointed as her Assistant in 1960. Judita had been the best student of mathematics at the university for generations, or so the story goes (Nikolić-Despotović, 2004; 14). Her younger fellow-students, later professors at the University of Novi Sad, Irena Čomić and Danica Nikolić Despotović, remember that students had great respect for professors but also some kind of fear for them. Despite all efforts of professors to travel from Belgrade, they were not always accessible to their students. The professional literature in Serbian language was still insufficient at that time, and students could not use foreign titles because their knowledge of English, French, German or Russian was modest. The only person who was able to answer at any moment a variety of questions by curious students, was Judita Cofman. As her knowledge of foreign languages was high, she was almost the only one among the students who could use German, English and Russian textbooks and widen her knowledge of mathematics, which she used to unselfishly share with her colleagues. They felt that she knew all there was to know about mathematics! As soon as she was made an Assistant teacher, in collaboration with students she published the lecture notes on Ruler-and-compass Constructions. This was the first publication in the field of mathematics issued at the Faculty of Philosophy in Novi Sad and as such heralds what was to become an abundant mathematics publishing activity at the University of Novi Sad.

4 Mathematische Zeitschrift, Archives Mathematica, Canadian Journal of Mathematics, for example (Nikolić, 2012: 97).

5 Colloquio Internazionale sulle Teorie Combinatorie (1973) held in Rome, Atti del Convegno di Geometria Combinatoria e sue Applicazioni (1970) held at the University of Perugia and the International Conference on Projective Planes - Dedicated to the memory of Peter Dembowski (1973) held at the Washington State University (Nikolić, 2012: 95).

The following year, in 1961, she left to undertake postgraduate studies in Rome. There she studied with well-known Italian mathematician Professor Lucio Lombardo-Radice (1916-1982). As Lombardo-Radice contributed to finite geometry and geometric combinatorics together with Guido Zappa (1915) and Beniamino Segre (1903-1977), and wrote and published important papers concerning the Non-Desargues Plane, Judita Cofman chose the same field of mathematics for her scientific work. In 1963 she returned to Novi Sad defending her PhD thesis under the title *Finite Non-Desargues Projective Planes Generated by Quadrangle*, she took the first doctoral degree in mathematical sciences from the University of Novi Sad. The committee for her thesis defense consisted of Lombardo-Radice and professors Mirko Stojaković (mentor) and Mileva Prvanović.

As the holder of the Alexander von Humboldt scholarship she spent 1964/65 school year at the University of Frankfurt/Main. The following six years she spent as a lecturer professor at Imperial College in London (University of London), where she was also engaged in her research work. In 1970 she was a visiting professor at the University of Perugia (Italy). From 1971 to 1978 she taught mathematics at the universities in Tübingen and Mainz. During this period she took part in three major conferences - International Colloquio on Combinatorial Theory held in Rome, Combinatorial Geometry and Applications held at the University of Perugia and the International Conference on Projective Planes held at the Washington State, which hosted all the important mathematicians of that time whose field of work involved Projective Planes.

From 1978 till 1993 she worked at the state Putney High School in London and became interested in the teaching and methodology of mathematics at secondary level. The academic year 1985-86 she stayed at St. Hilda's College, Oxford, during which she enjoyed the teacher fellowship. She spent these six years professionally absolutely dedicated to the pedagogical-methodological work with teachers of mathematics and talented students. Judita also taught on several advanced master classes held at the City of London School, together with teachers Terry Heard and Martin Perkins. In 1987 she was the member of training staff of the British Olympic team at 28th International Mathematical Olympiad held on Cuba.[6] In April 1993 she participated at Second German meeting of the European Woman in Mathematics in Tübingen with a talk *On the role of problem solving in math classes*. She collaborated with several foundations and associations (Sir John Cass Foundation, Advanced Royal Institution Mathematical Classes, Association of Gifted Children in Great Britain). She was organizer, and active participant and lecturer of several inspirational International summer camps for young mathematicians. The number of people that attended the maths camps was also coached by Judita in preparation for the Maths Olympiads (Jeroen Nijhof, Anders Bjorn and Alex Selby for example).

She was the editor of journal *Hypotenuse* which was populated by articles closely connected to her seminars at these camps. She also gave regular seminars in Germany for teachers of mathematics and students preparing to become teachers, as well as lectures within didactics seminars (Didaktik der Mathematik - Seminar der Universität Freiburg).[7]

6 The report by Mr. Robert Lyness, leader of the British team at 28th International Mathematical Olympiad, Cuba 1987 tells us of this occassion: «Our team was selected by means of the National Mathematics Contest and the British Mathematical Olympiad, followed by some postal tuition and a residential selection/training session which included a further test. This session was held at the Ship Hotel, Reading, from Friday 8th May to Sunday 10th May 1987. It was staffed by Judita Cofman, David Cundy, Terry Heard, John Hersee, Paul Woodruff, and myself. The training programme consisted of short lectures and tutorial periods during which the participants had opportunities to expound their own solutions to problems. It proved extremely helpful. All these activities are the responsibility of the Mathematical Association›s *National Committee for Mathematical Contests*.»

7 *Mathe mal anders*, Freizeitaktivitäten für Schüler und Studenten, lecture held on the 5th of December 1995.

From 1993 until her retirement in august 2001 she worked as the professor of didactics of mathematics at University of Erlangen, Nürnberg and became the head of the Mathematics Teaching Methods Department there (Prvanović, 2002; 57). In her spare time she conducted Maths-Workshops for 13-19 years old youngsters. About her experience gained during these Workshops it could be heard and seen from the video being recorded at the FAU College Alexandrinum (Collegium Alexandrinum) as a part of Projekt Uni-TV. Judita Cofman gave the talk *Mathematik macht Spass! - Über Workshops für Gymnasial schülerinnen und -schüler am Mathematischen Institut on 24th of June 1999*.[8] In September 2001 she was invited to participate in the work of the Postgraduate Studies Department for Mathematics Teaching Methods and appointed a professor at the Faculty of Natural Sciences at University Kossuth Lajos in Debrecen, Hungary, where she passed away on 19th December of the same year.

Work in the Field of Mathematical Education

The second period of Judita Cofman's life and scientific work, from 1980 till 2001, was marked by theory and practice in the field of pedagogy and didactics of mathematics dedicated to students and their teachers. In this scientific engagement there were no momentary ascents of mind and creation of new systems, no new theories nor proving theorems or conjectures; what mattered to her seems to had been an overall understanding of teaching mathematics, approach to students and different teaching methods through a well-balanced division between theory and problems designed to motivate students to think and work independently on solving them. Judita Cofman possessed all the preconditions to be successful in the methods of teaching mathematics she was a mature personality, her mathematical ability was proven by his publication record, and most importantly, she cherished the deep and true love for children and their learning.

Judita Cofman collaborated with a number of universities around the world in preparing future mathematical teachers, and had an intensive cooperation with mathematics institutes and various associations of mathematicians. She constantly emphasized the importance of the quality of teaching of mathematics, from the lower grades of elementary school to university level. She was known for being an exceptionally good teacher, who had a very responsible attitude towards her profession, which was the result of her great respect for her audiences and the science of mathematics, and she prepared thoroughly for her lectures. In 1984 Judita started an intensive collaboration with associations and methodological centers dedicated to teaching in the Hungarian language in Vojvodina. Her work is known in Hungary as well. She maintained a close contact with the universities in Budapest, Szeged and Debrecen in Hungary, and took part in the work of camps for talents and future teachers of mathematics.

The central problem of her engagement in didactics of mathematics was how to motivate pupils to think and work independently on their solutions even at an early age.[9] Judita Cofman thought that one way of introducing youngsters to independent study was to get them involved in work on projects. One such example was her well known International Camps for young mathematicians held in England and Germany (Cofman, 1990; preface and Cofman, 1986; Gardiner and Jones, 1985). The contents and organization of her well-known book *What to solve? - Problems and suggestions for young mathematicians*, is a compilation of problems and solutions discussed during seminars and sessions on problem solving in these camps. The organization of the text, selecting and grouping of questions, comments, references to related mathematical topics and instructions on teaching, represent the core of Judita's ideas on the teaching of mathematics. During the process

8 See the Web page http://www.university-tv.de/ca.html or directly the video of her talk http://giga.rrze.uni-erlangen.de/movies/collegium alexandrinum/ss99/19990624.mpg

9 Cofman (2000).

of solving problems and finding answer to the given question, campers-pupils and readers of her book were guided gradually step by step by encouraging independent thinking and researching, as well as through the variety of approaches to problem solving.

The atmosphere at the camp held in the summer of 1984, near Chelmsford in Essex is described well by Heather Cordell, at the time a fifteen-year-old girl from London.[10] Here it is her account:

"Like its predecessors, the camp in 1984 proved to be a resounding success. This was in no small way helped by the fact that its participants came from all over Europe, from nine different countries altogether, each bringing his/her own way of looking at mathematics and tackling mathematical problems. An average working day at the camp would start at 9:00 a.m. with a demonstration of problem-solving techniques. These problems varied from day to day in both standard and topic, so that a wide variety of interest and ability could be catered for. Examples included a proof of the existence of an infinite number of primes, ways of solving problems by the 'pigeon hole principle', and many others. Next came a session of project work. The projects were stimulated by particular problems (from a list provided) or were chosen by the students themselves. A practical interest in bell-ringing inspired one participant to investigate the various permutations possible on a certain set of bells. Many of the projects were a result of combined efforts; a more advanced student could often use his/her knowledge to work with a less advanced but equally dedicated student. In the second week, the students themselves led discussions about their results and difficulties with their projects. The first afternoon session lasted from 3:00 p.m. to 4:00 p.m., and consisted of a lecture a guest speaker or one of the tutors. The topics again varied, but usually involved some less traditional subjects such as codes and ciphers, non-conventional geometries, topology and ways to win *Nim*. The lecture contents would be expanded in the second afternoon session, from 5:00 p.m. to 6:00 p.m., to give more insight for the advanced participants. The fourth session was not compulsory - but many younger students did attend and enjoy what they could understand. Of course, there were many other opportunities for both relaxation and study. These included a visit to London (alternatively a country walk for those who preferred), an invitation watch the local bell-ringing and day spent in Cambridge, with two excellent lectures on *Convex sets and their applications in Economics* and *Algorithms*. All in all, everyone had a most enjoyable fortnight and is looking forward to participating again next year".

In her article *On the Role of Geometry in Contemporary Mathematical High School Education* (1996b), Judita Cofman listed remarks concerned primarily to geometry, based on her experiences with high school students:

1. In the contemporary education, with a curriculum overburdened with details from various fields of mathematics, there is a danger that the study of mathematics can stray into memorizing facts and a mechanical learning of algorithms. Contrary to this, the efforts should be directed at pupils' understanding the existing links between the phenomena they encounter in different fields of mathematical study. Geometry can play a certain role in such efforts, because the mathematical disciplines taught in high school are rich in details for which there are geometrical illustrations appropriate to the pupils' age. The application of such illustrations, on the one hand, facilitates the process of understanding of the totality of teaching material, and on the

10 Heather Cordell is today Professor of Statistical Genetics and a Welcome Senior Research Fellow in the Institute of Human Genetics at Newcastle University, UK. When she was once asked about who contributed to her becoming a scientist, among others, mostly professors and colleagues from University, she mentioned Judita Cofman as her high school mathematics teacher (See her text from 2003: Moving from Promise to Proficiency, *The Scientist*, 17 (8), 56.).

other, presents geometry as a science of an actual importance.

2. The importance of Euclidean geometry in teaching is supported by the fact that the shapes of this geometry are encountered in our living environment. The study of space is particularly facilitated by the study of solid geometry, which is, unfortunately, often neglected in syllabus and curriculum. There is an important fact in respect to geometrical features of space which is often forgotten; the majority of children possess a lot of elementary knowledge about objects, such as the cube or the sphere from the earliest age. This elementary knowledge can be extremely useful for introducing notions such as: defined and undefined elements, axioms and theorems, necessary and sufficient conditions, etc. All these notions are important for the field of mathematics while the familiarity with space can be used to make pupils grasp the essence starting from concrete examples.

3. Mathematics is one of the earliest scientific disciplines, an important segment of human cultural heritage. This fact must be reflected on the teaching of mathematics: it is advisable to draw pupils' attention, whenever an opportunity arises, to their historical background. The history of geometry is an important part of the history of mathematics, not only because geometry is one of the oldest branches of mathematics. The importance of geometry mostly lies in the fact that there were several major problems in this field, starting from the Ancient Greek age, which could finally be solved only in the 19th century. The solutions to these problems had been sought for ages; the attempts led to a series of new discoveries and contributed to a further development of the entire science of mathematics. One of the famous problems of geometry was the so called Delian problem of doubling the cube.

4. Teaching of geometry can also play a useful role in illustrating the achievements in the most current fields of mathematics.

5. The knowledge gained in the study of geometry can contribute to a better understanding of the phenomena from different fields of natural sciences.

6. For teaching of geometry to be successful, teaching personnel must have a solid knowledge of this subject. However, not only at schools, but also in university courses and other pedagogical institutions for training future mathematics teachers, there is a tendency of neglecting the study of geometry. This fact can lead to a drastic deterioration in the level of geometry teaching at schools. What is needed is an effort at elevating the respectability of geometry with the students of mathematics.

Conclusions

In her pedagogical work, Judita Cofman had the ability to raise simple mathematical truths onto a higher level and turn the elementary into a science. She knew historical genesis of each problem, where it originated from and how it was solved throughout history. She deemed that an important reason for teaching mathematics in schools was to promote independent pupils' thinking processes and powers of observation.[11] Throughout their schooling, pupils should be made aware of the links between various phenomena and they should be given the opportunity to discover these links on their own whenever this is possible. Moreover, pupils should be motivated to search for interdependence between seemingly unrelated topics. How can this be achieved? The key answer to the above question and generally to teaching of mathematics she gave in several papers and five books dedicated to mathematics teaching methods, which represent an outstanding approach to solving non-standard mathematical problems. Her historical approach to science and mathematical problems was the focus of her books which feature problems based on famous topics from the history of mathematics and a selection of elementary problems treated by eminent twentieth-century mathematicians.

11 Cofman (1998).

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др Александар М. Николић

Факултет техничких наука, Универзитет у Новом Саду

Дело Јудите Цофман у дидактици математике

Математичарка Јудита Цофман (1936–2001) рођена је у Вршцу 4. јуна 1936. године. Потиче из познате и некада богате породице Zoffmann, чији су се преци половином 18. века доселили из Немачке у Вршац. Иако пореклом Немци, постепено су усвојили мађарски идентитет, па се Јудита, обично, изјашњавала као Мађарица из Војводине. Припадала је првој генерацији ђака уписаних 1954. године на студије математике и физике на Филозофском факултету у Новом Саду. Била је најбољи студент, не само у својој генерацији већ и генерацијама после ње. На последипломске студије одлази 1961.

године у Рим. Тамо је студирала и учила код познатог италијанског математичара професора Лучија Ломбарда Радичеа. После две године проведене у Италији, враћа се у Нови Сад и брани докторат „Коначне недезарговe пројективне равни генерисане четворотемеником", и тако остаје упамћена као први студент који је докторирао на теми из математичких наука на Универзитету у Новом Саду. У комисији на одбрани њене тезе били су, осим Ломбарда Радичеа, и професори Мирко Стојаковић (ментор) и Милева Првановић. Њена докторска дисертација, као и њен целокупни научни допринос до краја седамдесетих година прошлог века, припадају области коначних пројективних и афиних равни, а њени радови у оквиру ове теме објављени су у престижним међународним математичким часописима. Са Филозофског факултета, али и из своје земље, одлази 1964. године – прво на Империјални колеџ Лондонског универзитета (Велика Британија), од 1970. године гостујући је професор на Универзитету у Перуђи (Италија), а од 1971. до 1978. године предаје математику на универзитетима у Тибингену и Мајнцу (Немачка). Почетком осамдесетих година прошлог века почиње да се интересује за проблеме наставе математике и више неће написати ниједан научни рад из претходне области којом се тако успешно бавила. Циљ нашег рада јесте да се скрене пажња на допринос Јудите Цофман у области педагогије и дидактике и методике математике кроз рад са младим математичарима, чему је у потпуности посветила други део свог живота и научних активности. Она је поседовала све предуслове да буде успешна и у методици и настави математике јер је била формирана личност, доказани математичар, и што је најважније, неговала је дубоку и истинску љубав према деци. Централни проблем њених активности у области дидактике математике био је како мотивисати ученике да размишљају и на својим решењима раде самостално већ у раном узрасту. У свом педагошком раду Јудита Цофман је имала способност да подигне једноставне математичке истине на виши ниво и претвори елементарне чињенице у зрелу науку. Знала је историјску генезу сваког проблема, где је настао и како је решен кроз историју. Сматрала је да је важан разлог за учење математике у школама промовисање независних процеса размишљања и моћи опажања код ученика. А један од начина за увођење младих у свет слободног и независног учења је њихово укључивање у рад на пројектима као што су били њени добро познати интернационални кампови за младе математичаре одржани у Енглеској и Немачкој. Садржај и организација њене чувене књиге „What to solve? – Problems and suggestions for young mathematicians", као компилацију проблема и решења, разматрани су током разних педагошких семинара као и при решавању проблема у тим камповима. Посебно се истичу њена размишљања и ставови о важности научне области, историје и наставе геометрије који су, на основу њеног богатог искуства као математичарке и професорке, формирани при раду са средњошколцима и талентованим математичарима. За њу је математика, као једна од цивилизацијских најранијих научних дисциплина, била важан сегмент људске културне баштине. Ова чињеница, сматрала је, мора се одразити и на наставу математике – препоручљиво је да се скрене пажња ученицима, кад год се укаже прилика, на историјске позадине области које се обрађују. Историја геометрије је важан део историје математике, не само зато што је геометрија једна од најстаријих грана математике већ и зато што је то прва логички и строго аксиоматски заснована област математике која се учи у школи. У септембру 2001. године Јудита Цофман била је позвана да учествује у раду Департмана за последипломске студије из наставе математике и именована је за професора на Природно-математичком факултету Универзитета Лајош Кошут у Дебрецину (Мађарска), где је преминула 19. децембра исте године .

***Кључне речи*:** Јудита Цофман, настава математике, дидактика математике.

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Original Paper

Mª Rosa Massa Esteve[1], PhD
Universitat Politècnica de Catalunya,
Centre de Recerca per a la Història de la Tècnica,
Barcelona, Spain

Historical activities in the mathematics classroom: Tartaglia's Nova Scientia (1537)[2]

***Abstract**: The History of Mathematics can be developed both implicitly and explicitly in the classroom. Learning about the history of mathematics can therefore contribute to improving the integral education and training of students. The aim of this paper is to analyze the proposal of an historical activity based on the work Nova Scientia (1537) by Tartaglia for use in the mathematics classroom. This analysis will show the use of a Renaissance mathematical instrument for measuring the height of a mountain in order to motivate the study of trigonometry in the mathematics classroom, as well as to show students the explanatory role of mathematics in regard to the natural world.*

***Key words:** History of mathematics, teaching, Niccolò Tartaglia, Nova Scientia, geometry.*

Introduction

The history of mathematics shows how mathematics has frequently been used to solve problems concerning human activity as well as for helping to understand the world that surrounds us. The study of historical processes enables us to see how the different aspects of mathematics have been combined together in a repeated interaction of application and development. Thus, for instance, geometry, which emerged as a means of measure, has evolved alongside the problems of measurement (Stilwell, 2010); trigonometry has developed in order to solve problems of both astronomy and navigation (Zeller, 1944), while algebra, which came more to the fore in problem-solving, especially in mercantile arithmetic during the Renaissance, was later to become an indispensable tool for solving problems in geometry and number theory (Bashmakova & Smirnova, 2000; Massa Esteve, 2005a). All this knowledge will undoubtedly enrich the mathematical background and training of teachers, some references to which can be found in the historiography (Calinger, 1996; Fauvel &Maanen, 2000; Demattè, 2006; Massa Esteve et al., 2011; Lawrence, 2012).

In Catalonia, the implementation of the history of mathematics in the classroom has for twenty years inspired some individual initiatives among

1 m.rosa.massa@upc.edu

2 This research is included in the project: HAR2013-44643-R.

teachers (Romero & Massa, 2003; Guevara et al., 2008; Roca-Rosell, 2011; Massa-Esteve, 2012). The academic year 2009-2010 saw the inauguration of a new course for training pre-service teachers of mathematics in secondary education. The syllabus of this Master's degree launched at the universities includes a compulsory section on the history of mathematics and its use in the classroom. One of the subjects of this course concerns engineers-artists in the Renaissance, and a proposal of an historical activity on this subject in the mathematics classroom has been presented to pre-service teachers.

The aim of this paper[3] is to analyze the proposal of the implementation of this historical activity, and also to discuss whether these kinds of activities can show students how mathematics may play an explanatory role in regard to the natural world. Furthermore, the paper considers whether working with instruments and following the procedures recommended to their users in the past offer students today a valuable appreciation of mathematical practices (Heering, 2012).

Usefulness of the history of mathematics in the classroom

The usefulness of the history of mathematics in the classroom is described through our theoretical and practical approach, with the aim of persuading people about the need for this type of training. Knowledge of the history of mathematics can assist in the enrichment of teaching tasks in two ways: by providing students with a different vision of mathematics, and by improving the learning process (Katz, 2000; Jankvist, 2009; Panagiotou, 2011).

A different vision of mathematics

Teachers with knowledge of the history of mathematics will have at their command the tools for conveying to students a perception of this discipline as a useful, dynamic, human, interdisciplinary and heuristic science (Massa Esteve, 2003, 2010). Teachers in possession of such knowledge are able to show students a further relevant feature of mathematics – that it can be understood as a cultural activity. History shows that societies develop as a result of the scientific activity undertaken by successive generations, and that mathematics is a fundamental part of this process. Mathematics can be presented as an intellectual activity for solving problems in each period. The societal and cultural influences on the historical development of mathematics provide teachers with a view of mathematics as a subject dependent on time and space and thereby add an additional value to the discipline (Katz & Tzanakis, 2011).

It is also worth pointing out that not only as teachers, but also as mathematicians, the history of mathematics enables us to arrive at a greater comprehension of the foundations and nature of this discipline. The history of mathematics provides the devotees of this science with a deeper approach to an understanding of the mathematical techniques and concepts used every day in the classroom. Knowing history of our discipline helps us explain how and why the different branches of mathematics have taken shape: analysis, algebra and geometry, their different interrelations and their relations with other sciences.

An improvement in the learning process

The history of mathematics as a didactic resource can provide tools to enable students to understand mathematical concepts better. The history of mathematics can be employed in the mathematics classroom as an implicit and explicit didactic resource (Jahnke et al., 1996).

The history of mathematics as an implicit resource can be employed by teachers in the design phase by choosing contexts, by preparing activities (problems and auxiliary sources) and also by drawing up the teaching syllabus for a concept or an idea. In addition to its importance as an implicit tool for

3 A first version of this paper was presented in the *First European Autumn School of History of Science and Education*, 15-16 November 2013 in Barcelona.

improving the learning of mathematics, the history of mathematics can also be used explicitly in the classroom for the teaching of mathematics. Although by no means an exhaustive list, four areas may be mentioned where the history of mathematics can be employed explicitly in Catalonia: 1) for proposing and directing research work at baccalaureate level using historical material; 2) for designing and imparting elective subjects involving the history of mathematics; 3) for holding workshops, anniversary celebrations and conferences, and 4) for implementing significant historical texts in order to improve understanding of mathematical concepts (Massa Esteve, 2005b; Romero et al., 2007, 2009; Massa & Romero, 2009). This paper is focused on the last point, which is, presenting an historical text involving mathematical instruments employed in the Renaissance.

Historical activities in the mathematics classroom

Historical texts can be used throughout the different steps in the teaching and learning process: to introduce a mathematical concept; to carry out an exploration of it more deeply; to provide an explanation of the differences between two contexts; to motivate study of a particular type of problem or to clarify a process of reasoning.

In order to use historical texts properly, teachers are required to present historical figures in context, both in terms of their own objectives and the concerns of their period. Situating authors chronologically enables us to enrich the training of students. Thus, students learn different aspects of the science and culture of the period in question in an interdisciplinary way. It is important not to fall into the trap of the amusing anecdote or the biographical detail without any mathematical content. It is also a positive idea to have a map available in the classroom to situate the text both geographically and historically.

Teachers should clarify the relationship between the original source and the mathematical concept under study, so that the analysis of the significant proof should be integrated into the mathematical ideas one wishes to convey. The mathematical reasoning behind the proofs should be analyzed and contextualized within the mathematical syllabus by associating it with the mathematical ideas studied on the course so that students may see clearly that it forms an integral part of a body of knowledge. In addition, addressing the same result from different mathematical perspectives enriches students' knowledge and mathematical understanding (Massa Esteve, 2014).

The aims of the implementation of the historical activity in the mathematics classroom are:

a) To learn about the sources on which knowledge of mathematics in the past is based;

b) To recognize the most significant changes in the discipline of Mathematics; those which have influenced its structure and classification, its methods, its fundamental concepts and its relation to other sciences;

c) To show students the socio-cultural relations of mathematics with politics, religion, philosophy and culture in each period, as well as with other spheres;

d) To encourage students to reflect on the development of mathematical thought and the transformations of natural philosophy.

Case study: Historical activity based on Tartaglia's *Nova Scientia* (1537)

The following historical activity deals with the work *Nova Scientia* (1537) by Niccolò Fontana Tartaglia (1499/1500-1557). In order to implement the activity in the classroom, it is recommendable to begin with a brief presentation of the epoch, the Italian Renaissance, and Tartaglia himself. The aims of the author as well as the features of the work would then be analyzed, and finally students are encouraged to construct an instrument for measuring degrees and to follow the reasoning of a significant proof, in order to acquire new mathematical ideas and perspectives. This classroom activity would be implemented in the last cycle of compulsory education (14-16 year olds) with the aim of introducing and motivating the study of trigonometry.

The context: The Italian Renaissance

The period from the mid-14th century to the beginning of the 17th century was the age of the Renaissance, so called because it represented the rebirth of interest in the Greece and Rome of Classical antiquity (Rose, 1975; Hall, 1981).

Artists, writers, scientists, and even the more refined craftsmen looked to the past for inspiration and examples on which to model their own work. Latin and Greek were the indispensable keys to style, knowledge, and good taste, assuming a foundational significance in education that they were to retain for centuries. This was the period of the great voyages of discovery which enlarged the horizons of the Western civilization, as did the invention of printing, with its incalculable effects upon human communication and the spread of information. The stream of wealth from the New World helped to develop the already growing economies of Europe. The major influence Renaissance had on technology was in the field of architecture. The abandonment of Gothic forms by the Italian architect Filippo Brunelleschi (1377-1446) and his successors, and the gradual spread of the Neo-Classical Palladian style of building from Italy over the whole of Europe involved changes in building techniques.

Teachers could argue that the inventions of the modern world demonstrated its technological superiority: this was especially the lesson of Jan Stradanus' *Nova Reperta* (1570), a volume of splendid engravings also produced near the end of the 16th century. We can use this image of *Nova Reperta* to show all these advances to students (see Figure 1).

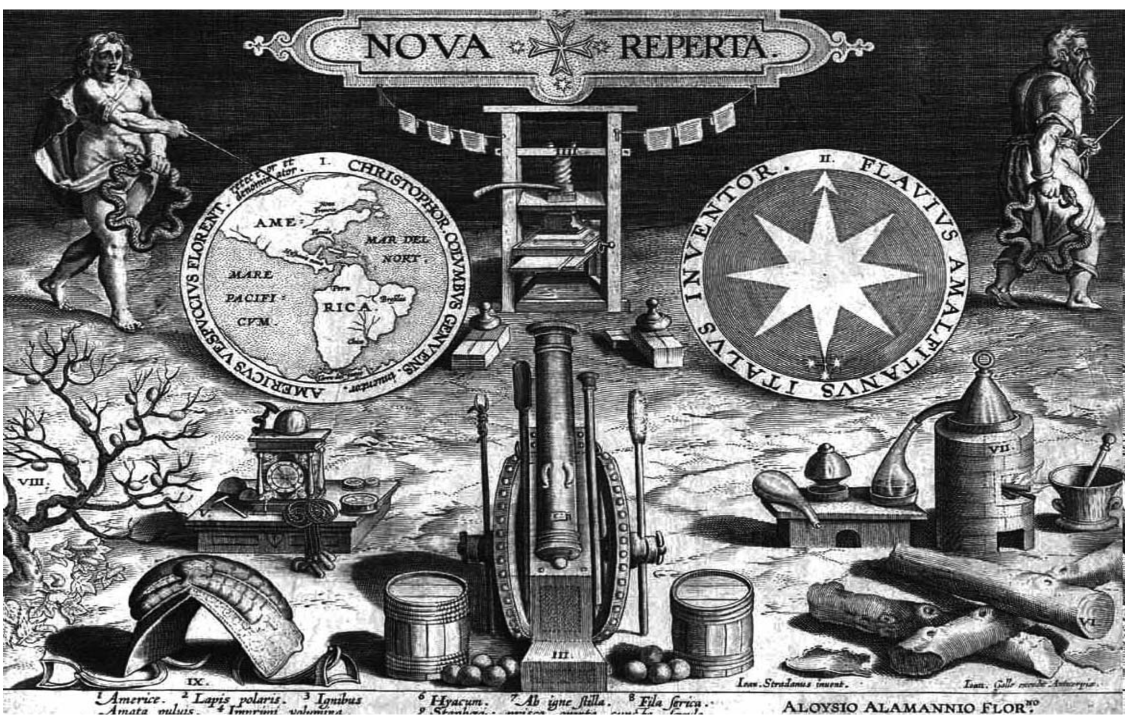

Figure 1. Nova Reperta

With this image the teacher can discuss with the students how the ancients had not mastered the "super-natural" force of gunpowder, nor discovered how to multiply books and pictures by printing. Neither had they found the direct route to the East, nor the New World to the West; they remained ignorant of the use of the magnetic compass and of other navigational aids which had made the 15th century voyages of discovery possible. The ancients also lacked windmills, iron-shod horses, the art of making spectacles, mechanical clocks and iron-founding.

In terms of the basic inventions and improvements made in the middle Ages, the Renaissance did little more than increase their size and scope. Machines became larger and more intricate and production increased. There were three major innovations during the Renaissance: gunpowder, the compass and printing. However, the Renaissance gave rise to a frame of mind which was increasingly receptive to further technological development.

The historical author: Tartaglia

Tartaglia, an engineer and scientist of the Renaissance, was taught first in abacus school and then further taught himself mathematics. Tartaglia belonged to that group of engineers and mathematicians who looked upon Archimedes as their role-model. Theory, practice, and knowledge and its application were all part of the goal of scientific knowledge of a mathematician. Hence, Tartaglia took a new role and presented a new image of the science of mathematics, which encompassed all these fields of study and action (Bennett & Johnston, 1996). Some works by Tartaglia are: *Nova Scientia* (1537, 2nd edition 1558), *Quesiti et Inventioni Diverse* (1546), *General Trattato di numeri et misure* (1556-1560) and *Euclid's Elements* (1543).

Tartaglia is deemed as a great mathematician of this period because of his use of geometry, and for his invention and development of a procedure for solving the cubic equation, but which Cardano later published claiming it as his own (Giusti, 2010; Gavagna, 2010). Tartaglia embodied the image of the engineer mathematician that appeared in Italy in the *Cinquecento* and whose aim was "to solve the problems of his professions and to practice the art of invention".

The historical work: Nova Scientia

In his work *Nova Scientia* (1537), Tartaglia introduced a new science: ballistics. In this work he tried to determine the form taken by the trajectory of a cannonball (Valleriani, 2013; Tartaglia, 1998).

In the frontispiece of the work dealing with the theory of ballistic phenomena, Tartaglia presents an image that seems to go back to the Platonic idea, according to which mathematics constitutes the key to the door of science and philosophy. The image depicts two fortresses: One is situated on a top of mountain or hill entitled Philosophy, and flanked by Plato and Aristotle; the other is situated at the bottom, and called the *Quadrivium,* which of course consists of Music, Arithmetic, Geometry and Astronomy but to which is added a new science: Perspective. Tartaglia is seen at the center as Master of Ceremonies, presenting the principles of science that constituted ballistics. To enter this *Sancta Santorum* of Knowledge one must pass through a door guarded by Euclid. Euclid's *Elements* in the *Cinquecento* period were not only the foundation but the paradigm or manner to attain all wisdom (propaedeutic function) (see Figure 2).

Tartaglia's book is not a treatise on motion in a medieval sense, that is to say, he does not analyze the nature of motion (Tartaglia, 1998). He states that he will address the study of the movement of a projectile ejected from a cannon or by whatever "artificial machine or matter that will be appropriated to throw violently a body equally weighty into the air." (Definition XIII). This current of thinking in, which including the artificial machine into the theoretical investigations, came to the fore in the middle of the XV century (Gessner, 2010). The practice was established by engineers and others trained in the ateliers of craftsmen from the north of Italy and Germany. The machine and its artifices were regarded as a way of conducting research into the world. In fact, the mechanics of the *Cinquecento* may be regarded as a science of machines. The theoretical analysis of the functioning of machines and their effects is predominant in other subjects in mechanics, but can use geometry because machines such as the balance, the lever and the pulley are simple to analyze by geometric methods.

Figure 2. Frontispiece of the Nova Scientia. Tartaglia, 1537.

At that time, the principal problems in the analysis of movement of a cannonball were the questions of what happened when the ball was in the air. These were: By how many degrees should the cannon be inclined to the horizontal so that the ball could hit a target located at a particular distance? At what inclination must a ball be fired so that the expected distance would be the maximum of possible distances?

Tartaglia, who was at the time professor of mathematics at Venice, gave the first answers to these questions. He asserted that the maximum distance of a ball fired from a cannon could be obtained by inclining the cannon 45º on the horizontal. Furthermore, he provided another answer that was even more surprising; Tartaglia claimed that the trajectory that the ball described through the air consisted of a curve. This claim contradicted the Aristotelian doctrine of movement, according to which the movement that the ball must follow will be a straight line until it reaches its maximum height, after which it will fall vertically to the center of the earth (Henninger-Voss, 2002). Thus, Aristotle's doctrine provided for no curved movement. However, the trajectory of a cannonball according to Tartaglia was composed of three parts; one rectilinear, one curve that follows an arc of circumference, both representing the trajectory's violent motion and, finally, one rectilinear of natural motion (see Figure 3).

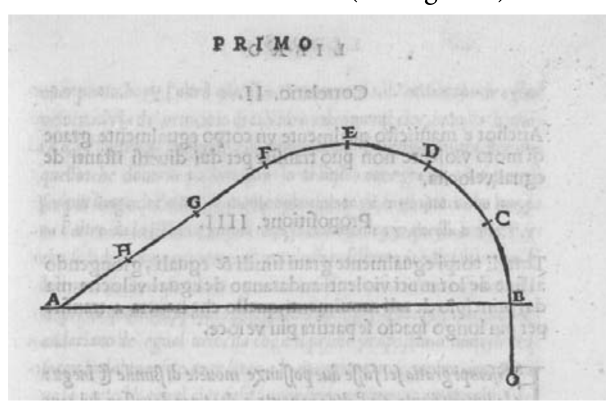

Figure 3. Tartaglia's movement. Tartaglia, 1537.

This work by Tartaglia enjoyed considerable success. By 1583, the text in Italian had reached seven editions and had been translated into many languages. Tartaglia, an expert on the matter, subsequently returned to the problem of movement in his work *Quesiti et invention diverse* (1546).

The text: the Euclidian way

Nova Scientia consists of 3 books. The first book contains 14 definitions, 5 suppositions, 4 commune sentences and 6 propositions with some corollaries. Book two contains 14 definitions, 4 suppositions and 9 propositions with some corollaries, and the third book contains 5 definitions and 12 propositions. In looking at Tartaglia's *Nova Scientia*, teachers in the classroom may comment with the students on the Euclidian way of presenting this new practical science (Ekholm, 2010).

In Proposition I of the first book, Tartaglia states his results on movement, including the following words: "Proposition First. All bodies of equal weight with natural movement, the further they move away from the beginning and approach their end point, the more they gain in speed."[4]

In the fourth supposition of the second book, he again addresses the inclination of the cannon on the horizontal to achieve the maximum distance. Teachers can discuss with students the inclination at which the expected distance would be the maximum (see Figure 4).

In addition, in the Proposition VIII of the second book he proves that the inclination of the cannon must be 45º, basing his reasoning on the proposition VII, where he uses geometry in similar triangles (see Figure 5):

> "Proposition VIII. If the same moving power ejects or throws equally heavy bodies, which are similar and equal to each other, violently through the air but in different manners, the one [equally heavy body] that accomplishes its transit at an elevation of 45 degrees above the horizon produces its effect farther away from its beginning and above the plane of the horizon than [if it were] elevated in any other way."[5]

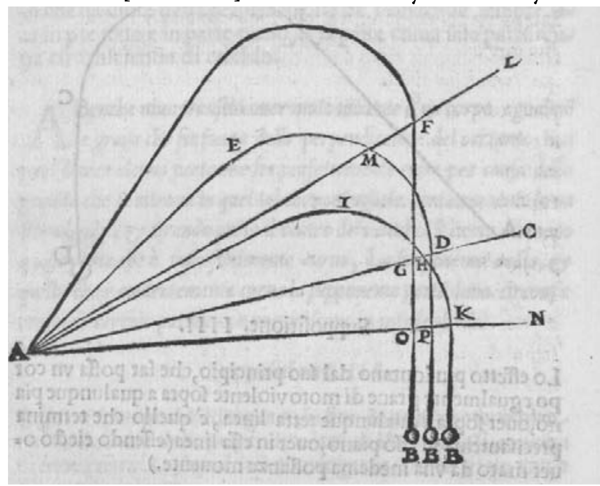

Figure 4. Tartaglia, 1537.

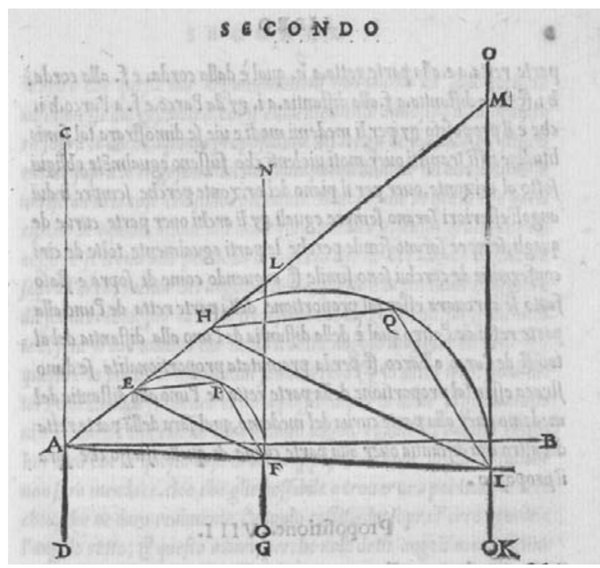

Figure 5. Figure of the Proposition VII. Tartaglia, 1537.

4 The translations are adapted from the English version by Valleriani of Tartaglia's edition of the 1558 (Valleriani, 2013)."Propositione Prima. Ogni corpo egualmente grave nel moto naturale, quanto piu el se andara aluntanando dal suo principio, over appropinquando al suo fine, tanto piu andara veloce." (Tartaglia, 1537).

5 „Propositione VIII. Se una medema possanza movente eiettara, over tirara corpi egualmente gravi simili, et eguali in diversi modi violentemente per aere, Quello que fara il suo transito elevato a 45 gradi sopra a l'orizonte fara etià il suo effeto piu lontan dal suo principio sopra il pian de l'orizonte che in qualunque altro modo elevato" (Tartaglia, 1537)

The mathematical instrument

Tartaglia constructs two gunner's quadrants, one with a graduate arc to measure the inclination of the cannonball, and the other instrument for solving the problem of measuring the distances and height of an inaccessible object. He offers an explanation of the first instrument at the beginning of the book in the dedicatory letter, as well as examining its construction accurately. He also gives examples with cannon (see Figure 6).

Figure 6. Dedication letter. Tartaglia, 1537.

In the third book, from the Proposition I to the Proposition IV, he provides a description of the material required for constructing the second gunner's quadrant: the rule and the setsquare, and checks its angles in the following propositions. Finally in the Proposition VI of the third book, Tartaglia constructs this gunner's quadrant (see Figure 7).

This gunner's quadrant is used by Tartaglia for measuring the height of inaccessible objects in the propositions of the third book, as shown below.

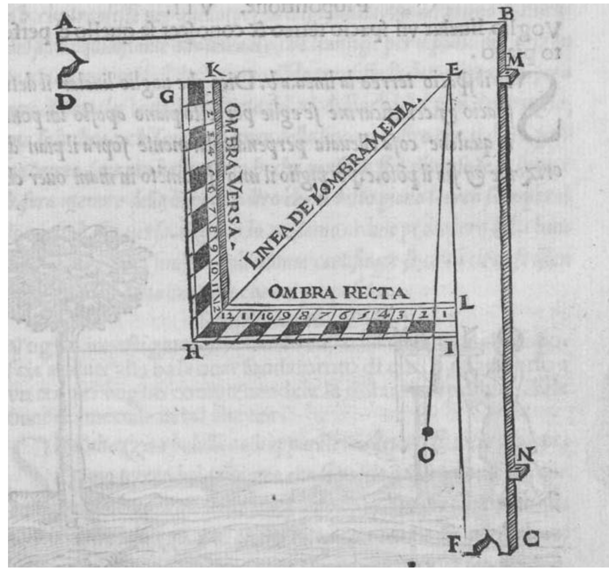

Figure 7. The gunner's quadrant. Tartaglia, 1537.

The significant proof: Proposition VIII of the third book

The proposition we will look into more detail I find a good example to be employed in the classroom because Tartaglia uses the gunner's quadrant, while at the same time using geometry in similar triangles in the proof to determine the distances and height of an inaccessible object. In the classroom implementation, students could be prompted to reproduce the reasoning of this proof with the geometry of triangles before introducing the trigonometry.

In the Proposition VIII of the third book Tartaglia proves how to obtain the height of a visible, but inaccessible object. He claims:

> "I would like to investigate the height of a visible object that one can move to the level of the base, and at the same time I would like to determine the distance through the hypotenuse or diameter of the height." [6]

6 "Propositione VIII. Voglio investigare l'altezza de una cosa apparente che si poscia andaré alla basa, over fondamento di quella, etiam tutto a un tempo voglio comprehendere la distantia ypothumissale, over diametrale di tal altezza". (Tartaglia, 1537).

The image of this proposition clarifies the geometric reasoning (see Figure 8):

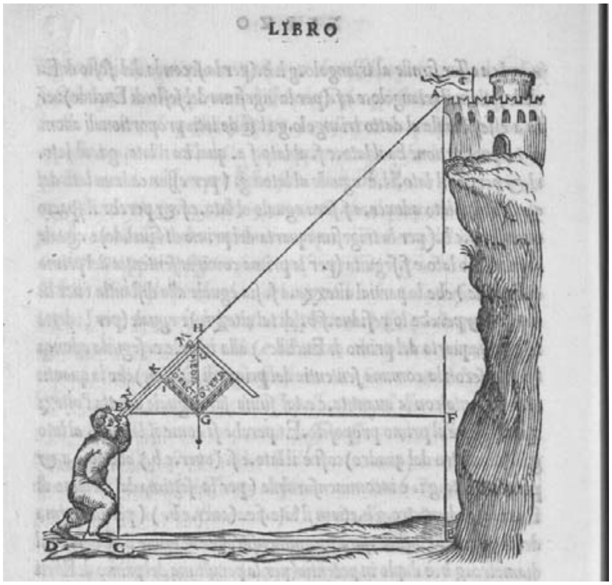

Figure 8. Figure of the Proposition VIII. Tartaglia, 1537.

After providing accurately an explanation of the construction of gunner's quadrant, together with the students the teacher could follow the reasoning of the proof using the similarity of triangles. For example, they can draw a figure with triangles that reproduces the geometric problem (see Figure 9).

Together with the students, the teacher can reproduce the geometrical proof using similar triangles, Pythagoras' theorem and Thales' theorem.[7] The teacher can also show the use of this figure to solve other problems in the classroom; for instance, the height of a house, or a distance of an object. In fact, these kinds of problems are solved today by trigonometry, and furthermore this historical activity also justifies the introduction of the teaching of trigonometry.

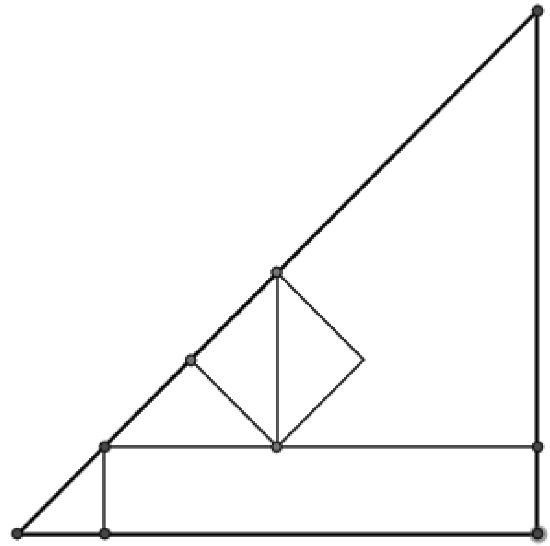

Figure 9. Reproduction of the mathematical problem

Concluding remarks

In order to transmit to the students the idea that mathematics is a science in a continuous state of evolution, and that it is the result of the joint and ongoing work of many people rather than knowledge amassed by independent contributions arising from flashes of inspiration, it is recommendable to present historical activities in the classroom. This historical activity shows the process from geometry to trigonometry and also how a mathematical instrument can be used in the mathematics classroom, the gunner's quadrant, for instance, or an instrument for measuring degrees, in order to obtain the height of inaccessible objects such as trees or mountains.

As regards to the question posed about whether working with instruments and following the procedures recommended to their users in the past can provide students today with a valuable appreciation of the past practices, it should be taken into account that when working with instruments

7 In this case I am referring to *Elements* VI. 2: "If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally; and, if the sides of the triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle."(Heath, 1956).

in the classroom, it may not be appropriate to follow exactly the instructions of the users in the past. The function and efficacy of these instructions of the past practices sometimes are not connected to the nowadays world of students. However, the replication of such procedures in the construction of instruments could inspire ideas for constructing similar instruments to reproduce this practice with students today. Students can learn about how mathematical instruments were used in the past to solve real problems.

Actually, in the Renaissance, technological developments in military and artistic spheres, as well as in scientific instruments, were made through the study of mathematics, which became increasingly regarded as a universal tool for solving problems. Thus, the question whether mathematics acquired an explanatory role in regard to the natural world gives rise to further questions about how and why this was so, and leads to discussions on the nature of mathematics. One may consider that this historical activity clearly shows the explanatory role of mathematics for solving problems of practical geometry in the Renaissance, such as the problem of the inclination of a cannon when one wishes to achieve the maximum distance, as well as the problem of finding the height of inaccessible objects. The mathematical ideas used for the proofs of these propositions can be found in Euclid's *Elements*: Pythagoras' theorem, Thales' theorem, and in the principles relating to incommensurable lengths. In fact, history of mathematics shows that mathematics is used to address a natural phenomenon, and in this sense it shows the usefulness of mathematics for revealing the natural world. The originality of this historical activity resides in the use of a text, which does not consist entirely of pure mathematics, to give an explanation for the movement of the military projectiles, which could be described as geometrization of real-life problems.

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др Марија Роза Маса Естеве
Политехнички универзитет Каталоније, Истраживачки центар за историју технологије, Барселона, Шпанија

Историјске активности на часовима математике: Тартаљина *Nova Scientia* (1537)

Историјски садржаји математике могу да се развијају и имплицитно и експлицитно на часовима. Учење о историји математике може да допринесе побољшању интегралног образовања и оспособљавања ученика. Историју математике, као имплицитни извор, наставници могу да користе да би осмислили фазу часа користећи различите контексте, припремајући наставне активности (проблемске ситуације и помоћне изворе за сазнавање) и креирајући наставни силабус у функцији формирања појмова или идеја. Осим као имплицитном средству за побољшање учења математике, историја математике може да се користи експлицитно у разреду ради поучавања математике. Имплементација важних историјских текстова може да обезбеди средства која ће ученицима омогућити да боље разумеју математички појам. Циљеви имплементације историјске активности на часовима математике су: а) учење о изворима на којима се заснива знање математике у прошлости; б) препознавање најзначајнијих промена у математичким дисциплинама – оне које су утицале на структуру и класификацију, на њене методе, основне појмове и везу са другим наукама; в) указивање ученицима на социокултурну везу математике и политике, религије, филозофије и културе, у сваком периоду, као и везе са осталим сферама, и коначно, што је најважније, подстицање ученика да се изразе у вези са математичком мишљу и трансформацијом природне филозофије. Циљ овог рада је анализа студије случаја предлога историјске активности, базиране на раду „Nova Scientia“ (1537) Никола Фонтане Тартаље (Niccolò Fontana Tartaglia (1499/1500–1557)), за коришћење на часовима математике. Ова анализа ће показати употребу ренесансног математичког инструмента за мерење висине планине да би се мотивисало проучавање тригонометрије на часовима математике, као и показивање улоге математике у објашњавању природног света. Штавише, у раду разматрамо да ли рад на инструментима и мерења помоћу њих, препоручиваних корисницима у прошлости, омогућавају ученицима у садашњости адекватно вредновање мерењем инструментима из прошлости.

Кључне речи: историја математике, поучавање, Николо Тартаља, „Nova Scientia“, геометрија.

Teaching Innovations, 2014, Volume 27, Issue 3, pp. 127–129

Book Review

SPIRITUAL LIFE ENRICHED BY NEW KNOWLEDGE

Mirko Dejić (2013). Number, Measure, Immeasurability.

From Mathematics to Anthropology (in Serbian).

Belgrade: Teacher Education Faculty, 337. p.

Two previous books by the same author (from 1990 and 1995) and his scientific and professional papers published between 2001 and 2012 in various periodicals and annuals form the basis of this book. Nevertheless, it cannot be said that this book represents the collection of the chosen papers or anything similar. It is the result of the serious effort of the author to give a critical review on the mentioned books and papers, to systematize them, and as far as it is possible, to make them comprehensible to wide scope of readers. The author, Mirko Dejic, particularly stressed their general cultural significance, and the strictly professional component became secondary issue.We are sure hat in this way, the author gave his contribution to positioning Mathematics as an integral part of general culture, its position belonging undoubtedly to it along the significant periods of the development of humankind. Nowadays this position is being impugned, although in the conditions of IT development and fast technological growth, it should be otherwise. Apart from this, a significant part in the book is devoted to the status of Mathematics in reality of Serbia, general and educational, as well as presence and influence of Serbian Mathematicians and other creators in Mathematics development and its applications worldwide.

Texts in the books are formed within three chapters: *I Philosophy of Mathematics; II History of Mathematics; III Mathematics and Religion.* We are going to try to reveal the contents through brief reviews.

At the very beginning of the first chapter, in the sub-chapter Mathematics, there is a short review of efforts of Mathematicians, and not only them, to come to rational, widely accepted definition of Mathematics as science and universal actions of a sensible being as the man is. The reader, reading many definitions, by the rule adjusted to the needs or subjective relation of the individual, will see that generally accepted definition does not exist. We are going to be free to add to the definition a sentence written by a famous German-American Mathematician Richard Courant (1888–1972)) (paraphrasing): "The active doing of mathematics will help us to get the answer to the question: What is mathematics?". In the second chapter, *Nature of Mathematical Knowledge,* the author deals with the questions of building deductive mathematical systems. With the example of axiom based geometry, from Euclidean to non-Euclidean, he points at the significance of creators of the non-Euclidean geometry from traditional loyalty to perception experience, which resulted in significant encouragement to creators in Mathematics and other scientific fields to found new scientific disciplines, and build up new theories. This is how, for instance, in theoretical physics, theory of relativity and quantum physics appeared. In the chapter *Nature of mathematical being,* a review of some philosophical learning (Platonism, constructivism, intuitionism, nominalism, realism and formalism) is given, first of all through description of bases of attitudes about origins and essence of mathematical objects and relations between them. The third chapter is *Mathematical creativity.* The significance which M. Dejic gives to this chapter, the way it is being discussed, the scope and attention relating to the issues in question, we can recognize his life vocation towards revealing and nourishing the gifted young mathematicians, their careful leading to the level of knowledge and devotion to Mathematics, in which their mathematical abilities will enable them to create within its frames. Carefully chosen examples illustrate various aspects of the way from spotting the problem to enlightment, through which discovery appears and in this way to the realization of creators' tendencies.

The second chapter starts with the sub-chapter *Short review on the historical development of Mathematics until Dekart,* within which M. Dejic, accepting periodisation of the historical development of Mathematics of the eminent Russian mathematician A. N. Kolmogorov (Андрей Николаевич Колмогоров, 1903–1987), showed development of Mathematics through the period of its foundation, development of elementary Mathematics, through the Mathematics of variable quantities and the period of contemporary Mathematics. In the next chapter, this review became richer because of the biographies and works of antique mathematicians: Tales, Pythagoras (Πυθαγόραζ), Plato (Πλατων), Euclid (Ευκλειδηζ), Archimedes, Eratosthenes, (Ερατοσθενηζ), Heron (Χερον), Diofant (Διοφαντοζ)... The next two chapters are devoted to Serbian Mathematics, its highlights in forming Serbian Mathematics School in the 19th century, as well as great contribution of Mihajlo Petrovic Alas (1868–1943), the most significant of its members, for teaching Mathematics in high schools in Serbia. Further, on, within this chapter, the contents are analysed, concerning the history of some mathematical symbols and terms, developmental phases of the concept of numbers, their names and ways of noting them. Chapters, which follow, are about the counting tools (abacus and tables) and counting with the aid of them. The last subchapter of this chapter refers to number systems, history of numeration from its existence to contemporary numeration the origin of the zero and its marking through history, as well as ways of noting big numbers. The reader will, in the way in which final subchapters are analysed, recognise interest of the author, his deep and thorough knowledge of those contents, which undoubtedly comes from his scientific and professional interests and realised results from this area.

The third chapter will provoke significant interest of wide scientific and professional public. The title *Mathematics and religion* points out its interdisciplinarity, as well as non-standard, and in scientific and professional works of Mathematicians rarely present contents. Being aware of these facts, the author opens a new chapter with the sentences in which he stresses that Mathematics and religion, although at first sight have nothing in common, are close, even interwoven in many segments; [...]." The chapter *Mathematics in Religion and Religion in Mathematics* starts by stressing similarities between Mathematics and Religion. It seems to us that these similarities have been successfully expressed through the role of intuition through sensing the truth and the need to approve this truth, both in Mathematics and Religion. Nevertheless the methods of approving are different. There is a parallel between dogmats, which are a part of dogma, who present the basis of religion and adopt it without checking, they are trusted, and the system of axioms, which have the same role in forming each of mathematical theories. The guarantee of the axiom truth is the man's mind and the guarantee of the truth of dogma is God. Various mathematical proofs about the existence of God and the fact that these tendencies are met in the work of mathematicians who created Mathematics through their work, witness the presence and justification of stressing the mentioned parallel. The author states the example of the genius Indian Mathematician Ramanujan (Srinivas Ramanujan), who stated at the beginning of the 20th century that in his dreams he received visions from Gods in the from of complex mathematical truths. In this way, he gave to Mathematics significant results, incomprehensible to the mind. The chapter finishes with reviews of the influence of religion to forming some mathematical terms, relation of churches towards Mathematics, presence of Mathematics in the Bible and review on the relation between Mathematics and religion. In the chapter *Influence of Religion on Development of the term Infinity,* the author gave the retrospective of development of the term infinity in Mathematics, which analogues in religion can be recognized in the terms infinity and immensity, which can be found in the Bible. The author sees many difficulties and challenges concerning introduction and using this term. In the book, the reader will face Aristotle's (Αριστοτεληζ) problem of actual and potential infinity and Zenon's paradox and Euclidus's (Ευκλειδηζ) theorem about the infinity of the set of simple numbers and Cantor's Канторове (Georg Cantor) transfinite numbers and his statement that he is only the God's messenger and famous Kronecker's (Leopold Kronecker) statement that whole numbers were made by God, and everything else is made by man". The chapter *Mystique of Numbers* in the first part leads the readers to Pythagorean school, in which numbers and their relations represent the essence of the real world, through their world of polygonal numbers and introduction of different classes of numbers, with the stressed mystic abilities, as well as the crisis they faced by the realization that the side and the diagonal of the square are not co-measurable. Certain knowledge about the mystique appearances and the role of Mathematics in oculistics will certainly attract readers. It is particularly seen in the titles of sub-chapters *Numbers determine human character, Destiny is in names* or *How to determine a fortunate city? Magic of Number 7…* Up to the final sub-chapter *Aritmology of Early Christian Scholars* that can briefly be characterized by the attitude of Nicomah I (Νικομαχ I), (1st century BC): "Everything in nature is determined or is in accordance with number, according to thoughts and mind of the one who created it". The chapter *Mathematicians Priests, Monarchs and Meolosts* is the result of many and various kinds of pursue of the author. The chapter starts with the review of life and mathematical contributions of about forty priests and theologians who come from the environment in which we can put Western civilization (Ancient Rome, Italy, England, France, Germany, Spain…). We can see many famous names among them. We are going to state some of them, without any pretentions to estimate their contribution to Mathematics. Those were Roger Bacon, Bernhard Bolzano, Rudjer Boskovic, Bonaventura Cavalieri and Lucca Paciolli, Michael Stifel. M. Dejic included in his book the chapter *Orthodox Monarchs – Mathematicians* too. The life and work of the Russian Monarch from the 12th century Kirik Novogorodski was presented, as well as Byzantine Monarcchs from the 13th and 14th centuries Maxim Planuedes, Theodore Metohit and Argir, Serbian Monarch Lazar Hilandarac who lived between 14th and 15th century and I.M. Pevusin, Russian priest from the 19th century. The writer of this review was particularly impressed by the new "meeting" with the first, up to now preserved Russian manuscript of mathematical contents of Kirk Novogorodski (Кирик Новгородский), what he saw in the ancient Orthodox Monastery Veliki Novgrod. In the chapter *The First Mechanical Clock in Moscow, a piece of the Serb Lazar Hilandarac*, a short history of medieval mechanical clocks is given, as well as the historical situation in Byzantium and Russia of that time and the history of producing and technical characteristics of the clock which the Monarch Lazar Hilandarac made in Kremlin in 1404. Undoubtedly, this represents a significant detail within the frames of cultural history of the Serbs. In the chapter, *The System of Measurement in the Bible,* there is a list of measures for length, volume, weight and money made, which were afterwards transferred into the contemporary measurement system. The author made some additional remarks concerning the efforts of he translators of the Bible into Serbian, to find suitable words, sometimes introducing new words, enriching lexical fund of Serbian. In the final chapter, *Calculating the Date of Celebrating Easter* the author thoroughly and widely shows historical thoughts concerning these issues, including efforts for the calendar reform and the role of the Serbian Orthodox Church within them and Milutin Milanković. Apart from stating table for date determination of Easter, the author instructs the reader how to directly determine the date of Easter in certain year.

We are convinsed that the reader of the book ***Number, Measure, Immeasurability,*** *From Mathematics to Anthropology* by Mirko Dejic, will enrich spiritual life by new knowledge and that the new contents will motivate him/her towards new challenges.

Vladimir Mićić, PhD

Приказ

ДУХОВНИ СВЕТ ОБОГАЋЕН НОВИМ САЗНАЊИМА

Мирко Дејић (2013). Број, мера и безмерје.

Од математике до антропологије.

Београд: Учитељски факултет, 337 стр.

У основи ове књиге налазе се две раније књиге аутора (из 1990. и из 1995. године), као и његови научни и стручни радови публиковани у периоду од 2001. до 2012. године у разним часописима и зборницима радова. При томе се за ово дело не може рећи да представља зборник одабраних радова или нешто слично. Оно је резултат озбиљног труда аутора да се на садржаје поменутих књига и радова критички осврне, обједини их и систематизује, као и да их, у мери у којој је то оправдано и могуће, учини приступачним и разумљивим широком кругу читалаца. Аутор М. Дејић је настојао да посебно нагласи њихов општекултуролошки значај, што је ужестручну компоненту, по правилу, доводило у други план. Тиме је, уверени смо, дао свој допринос позиционирању математике као саставног дела опште културе, положаја који јој је неоспорно припадао током значајних периода у развоју човечанства, а данас јој се вишегласно оспорава, иако би у условима опште компјутеризације и бурног технолошког развоја морало бити обрнуто. Осим тога, значајно место у књизи посвећено је статусу математике у стварности Србије, општој и образовној, као и присуству и утицају српских математичара и других стваралаца у развоју математике и њених примена у ширим, светским размерама.

Текстови у књизи су оформљени у оквиру трију глава: *I. Филозофија математике; II. Историја математике; III. Математика и религија.* Покушаћемо да кроз кратке приказе приближимо читаоцима њихове садржаје.

На самом почетку прве главе, у поглављу *Математика*, дат је краћи преглед настојања математичара, и не само њих, да се дође до рационалне, широко прихватљиве дефиниције математике као науке и као универзалне делатности разумног бића, какав је човек. Читалац ће, читајући бројне дефиниције, по правилу прилагођене потребама или субјективном односу појединаца, уочити да таква опште прихваћена дефиниција не постоји. Дозволићемо себи слободу да наведене дефиниције употпунимо реченицом коју је написао познати немачко-амерички математичар Р. Курант (Richard Courant (1888–1972)) (парафразирамо): „Активно бављење математиком помоћи ће нам да дођемо до одговора на питање: Шта је математика?“. У другом поглављу, *Природа математичког знања*, аутор се, пре свега, бави питањима изградње дедуктивних математичких система. На примеру аксиоматског заснивања геометрије, од еуклидске до нееуклидске, он указује на значај отклона стваралаца нееуклидске геометрије од традиционалног робовања опажајним искуствима, што је резултирало значајним подстицајима ствараоцима у математици и другим научним областима да заснују нове научне дисциплине, изграде нове теорије. Тако су, на пример, у теоријској физици настале теорија релативности и квантна физика. У одељку *Природа математичког бића* дат је приказ неких од филозофских праваца (платонизам, конструктивизам, интуиционизам, номинализам, реализам, формализам), пре свега кроз приказ основа на којима се темеље ставови о пореклу и суштини математичких објеката и односа међу њима. Треће поглавље је *Математичко стваралаштво*. У значају који М. Дејић придаје овом поглављу, начину на који га обрађује, као и обиму и пажњи с којом се односи према проблематици о којој је реч, препознајемо његово животно опредељење према откривању и неговању обдарених младих математичара, њиховом пажљивом вођењу до нивоа знања и посвећености математици, на којем ће им њихове математичке способности омогућити да стварају у њеним оквирима. Пажљиво одабрани примери илуструју бројне аспекте пута од уочавања проблема до блеска (озарења), кроз који се стиже до открића и тиме до остварења стваралачких тежњи.

Другу главу отвара поглавље *Кратак осврт на историјски развој математике до Декарта,* у оквиру којег је М. Дејић, прихватајући периодизацију историјског развоја математике истакнутог руског математичара А. Н. Колмогорова (Андрей Николаевич Колмогоров, 1903–1987), приказао развој математике кроз период њеног рађања, период развоја елементарне математика, период стварања математике променљивих величина и период савремене математике. Овај је преглед у следећем поглављу употпуњен приказима живота и дела најзначајних античких математичара: Талеса (Θαληζ), Питагоре (Πυθαγόραζ), Платона (Πλατων), Еуклида (Ευκλειδηζ), Архимеда (Αρχίμηδηζ), Ератостена (Ερατοσθενηζ), Херона (Χερον), Диофанта (Διοφαντοζ), ... Следећа два поглавља посвећена су српској математици, њеним узлетима и формирању српске математичке школе у 19. веку, као и великом доприносу Михајла Петровића Аласа (1868–1943), најзначајнијег од њених чланова, настави математике у средњим школама у Србији. Даље су, у оквиру ове главе, обрађени садржаји о историјату неких математичких симбола и термина, фазама развоја појма броја, њиховим називима и начинима записивања. Следе поглавља о првим рачунарским помагалима (абакусима и таблицама) и рачунању помоћу њих. Завршна поглавља ове главе односе се на бројевне системе, историју нумерације од њеног настанка до савремене нумерације, порекло нуле и њено записивање кроз историју, као и записивање великих бројева. Читалац ће у начину обраде завршних поглавља друге главе препознати наглашено интересовање аутора, његово дубоко и свеобухватно познавање тих садржаја што, несумњиво, проистиче из његових научних и стручних интересовања и остварених резултата из те проблематике.

Трећа глава ће, сигурно смо у то, изазвати значајно интересовање широке научне и стручне јавности. Њен наслов *Математика и религија* указује на интердисциплинарност, као и на нестандардне, и у научним и стручним радовима математичара, и не само њих, ретко присутне садржаје. Свестан ових чињеница, аутор отвара ову главу реченицама у којима истиче да „Математика и религија, иако наизглед немају додирних тачака, у многим сегментима се додирују, чак и прожимају; [...].“ Поглавље *Математика у религији и религија у математици* почиње истицањем сличности математике и религије. Нама се чини да су ове сличности успешно изражене кроз улогу интуиције у наслућивању истине и потребу да се истина докаже, како у математици, тако и у религији. При томе се методе доказивања разликују. Постоји паралела између догмата, који чине догму, представљају основе вере и усвајају се без провере, верује им се, и система аксиома, који имају исту улогу у строгом заснивању сваке од математичких теорија. При томе је гарант истинитости аксиома човеков ум, а гарант истинитости догме је Бог. Бројни математички докази о постојању Бога и чињеница да таква настојања срећемо и у делима математичара који су својим делима стварали математику, сведоче о присутности и оправданости истицања поменуте паралеле. Аутор наводи и пример генијалног индијског математичара Рамануџана (Srinivas Ramanujan), који је почетком 20. века тврдио да је у сновима примао визије од богова у облику сложених математичких истина. Тако је дао математици изузетно значајне резултате, тешко схватљиве обичном човечјем уму. Поглавље се завршава приказима утицаја религије на формирање неких математичких појмова, односа цркава према математици, присуства математике у *Библији* и освртом на однос наставе математика и веронауке. У поглављу *Утицај религије на развој појма бесконачности* дата је ретроспектива развоја појма бесконачности у математици, чији се аналогони у религији могу препознати у појмовима вечности и неизмерности, присутним у *Библији*. Аутор се осврће на бројне тешкоће и изазове у вези с увођењем и коришћењем овог појма. У књизи ће читалац срести и Аристотелов (Αριστοτεληζ) проблем актуелне и потенцијалне бесконачности, и Зенонове (Ζενο) парадоксе, и Еуклидову (Ευκλειδηζ) теорему о бесконачности скупа простих бројева, и Канторове (Georg Cantor) трансфинитне бројеве и његово тврђење да је он само „Божји гласник“, али и познато Кронекерово (Leopold Kronecker) тврђење да је „целе бројеве створио Господ Бог, а све остало је дело људских руку“. Поглавље *Мистика бројева* у првом делу води читаоца кроз питагорејску школу, у којој бројеви и односи међу њима представљају суштину појавног света, кроз њихов свет многоугаоних бројева и увођење различитих класа бројева са наглашеним мистичним својствима, али и до кризе у коју су запали сазнањем да страница и дијагонала квадрата нису самерљиве. Одређена сазнања о мистичним појавама и улози математике у окултистици сигурно ће заинтригирати читаоце. Посебно ако се суоче с насловима потпоглавља типа *Бројеви одређују људски карактер, Судбина у именима* или *Како одредити срећан град?*, *Магија броја 7...* све до завршног потпоглавља *Аритмологија ранохришћанских мислилаца*, које се, укратко, може окарактерисати ставом Никомаха I (Νικομαχ I), (1. век наше ере): „Све што је у природи, одређено је и у сагласности са бројем, према предумишљају и уму онога који је све створио“. Поглавље *Математичари свештеници, монаси и теолози* резултат је обимних и свестраних трагања аутора. Поглавље започиње приказом живота и математичких доприноса четрдесетак свештеника и теолога који долазе из средина које, условно, можемо сврстати у западну цивилизацију (Стари Рим, Италија, Енглеска, Француска, Немачка, Шпанија...). Међу њима срећемо многа позната имена. Навешћемо нека, без претензија да процењујемо њихове доприносе математици: Роџер Бекон (Roger Bacon), Бернард Болцано (Bernhard Bolzano), Руђер Бошковић, Бонавентура Каваљери (Bonaventura Cavalieri), Лука Пачоли (Lucca Paciolli), Михаил Штифел (Michel Stifel). Посебно брижљиво М. Дејић је обрадио потпоглавље *Православни монаси – математичари.* Приказан је живот и дело руског монаха из 12. века Кирика Новгородског, византијских монаха из 13. и 14. века Максима Плануда (Maksim Planudes), Теодора Метохита (Theodor Metohit) и Аргира (Argir), српског монаха Лазара Хиландарца, који је живео на прелазу из 14. у 15. век и И. М. Первушина (И.М.Первушин), руског свештеника из 19. века. На писца овог приказа посебан утисак оставио је поновни „сусрет“ са првим, до данас сачуваним, руским рукописом математичке садржине Кирика Новгородског (Кирик Новгородский), који је имао прилике да види у просторима древног православног манастира Велики Новгород. У поглављу *Први механички сат у Москви, дело Србина Лазара Хиландарца* описан је историјат средњовековних механичких сатова, историјске прилике у Византији и Русији тог времена, као и историјат израде и техничке карактеристике сата који је монах Лазар Хиландарац направио 1404. године у Кремљу. То, несумњиво, представља значајан детаљ у оквирима културне историје Срба. У поглављу *Систем мера у Библији* направљен је попис мера за дужину, површину, запремину, масу и новац, које су, потом, преведене у савремени систем мера. Аутор је уложио додатни труд да би читаоцу приближио напоре преводилаца *Библије* на српски језик да пронађу одговарајуће речи, понекад уводећи и нове речи, чиме су обогаћивали лексички фонд српског језика. У завршном поглављу *Израчунавање датума празновања Ускрса* аутор исцрпно и свестрано приказује историјске чињенице у вези с том проблематиком, укључујући и напоре за реформе календара и улогу у њима Српске православне цркве и Милутина Миланковића. Осим навођења табеле за одређивање датума Ускрса, аутор обучава читаоца како да практично одреди датум Ускрса у жељеној години.

Уверени смо да ће читалац књиге ***Број, мера и безмерје**. Од математике до антропологије* аутора Мирка Дејића обогатити свој духовни свет новим сазнањима и да ће га упознати садржаји покренути у смеру нових изазова.

Проф. др Владимир Мићић

Professional information

FUTURE INTERATIONAL CONFERENCES

2015

- CERME 9, Prague, Czech Republic, February 4-8, 2015. http://www.mathematik.uni-dortmund.de/~erme/
- II International Symposium on Mathematical Education (SIME), Costa Rica, February 25-27, 2015. http://www.cimpa.ucr.ac.cr/simmac/en/sime.html
- The Second International Conference on Mathematics and Statistics, AUS-ICMS '15, The Department of Mathematics & Statistics, American University of Sharjah, April 2 – 5, 2015. http://www.aus.edu/ICMS15
- XIV CIAEM-IACME, Inter-American Conference on Mathematics Education, Tuxtla Gutierrez, Chiapas, Mexico - May 3-7, 2015. http://xiv.ciaem-iacme.org/index.php/xiv_ciaem/xiv_ciaem
- EARCOME 7 - ICMI East Asia Regional Conference on Mathematics Education, "In Pursuit of Quality Mathematics Education for All", Waterfront Hotel, Cebu City, Cebu, Philippines, May 11-15, 2015. http://earcome7.weebly.com/
- ICMI STUDY 23 conference, Macau SAR, China. Under the auspices of the International Commission on Mathematical Instruction (ICMI), University of Macau, Macau SAR, China, from June 3 to 7, 2015. http://www.umac.mo/fed/ICMI23/index.html
- ICTMT 12 - 12th International Conference on Technology in Mathematics Teaching Faculty of Sciences and Technology, University of Algarve, Faro, Portugal, June 24-27, 2015. http://ictmt12.pt/
- IV SIPEMAT, From June 29 to July 01, 2015, UESC, Ilhéus-Bahia, Brasil, Theme: Mathematical Education and contexts of cultural diversity. http://ppgemuesc.com.br/sipemat4/english/
- PME 39, 2015: July, 13-18: Hobart, Tasmania, Australia. http://www.igpme.org/index.php/annual-conference
- SEMT '15: International Symposium on Elementary Mathematics Teaching, Charles University, Faculty of Education, Prague, Czech Republic, August 16-21, 2015. http://www.semt.cz/
- Espace Mathématique Francophone, Alger, Algérie, 10-14 Octobre 2015. http://www.mathunion.org/icmi/events/details/?tx_ttnews%5Btt_news%5D=829&cHash=27033db6a2fd35703d9f546dec7fa48f

2016

- ICME-13, Hamburg, Germany, Sunday, 24th July to Sunday, 31st July 2016. http://www.icme13.org/
- PME 40, 2016: August, 3-7: Szeged, Hungary. http://www.igpme.org/index.php/annual-conference

Olivera Djokić, PhD
Teacher Education Faculty, Belgrade

Professional information

MATHEMATICS EDUCATION AND POPULARIZATION OF MATHEMATICS

Links to the videos of the invited talks and panels of the **"Mathematics Education and Popularization of Mathematics"** of ICM 2014 that was held in Seoul, Korea, August 13 - 21, 2014.

- ICM Panel 2. **How should we teach better?** http://www.youtube.com/v/Bvg2VgWu1p4

  Moderator

  Bill Barton, University of Auckland, New Zealand

  Panelists

  Bill Barton, University of Auckland, New Zealand

  Jean-Marie Laborde, Université Joseph Fourier, France

  Man Keung Siu, University of Hong Kong
- ICM Panel 1. **Why STEM?** http://www.youtube.com/v/WiHKMz-izNI

  Moderator

  Youngah Park, President of KISTEP (Korea Institute of S&T Evaluation and Planning)

  Panelists

  Jean Pierre Bourguignon, President of ERC (European Research Council)

  Ingrid Daubechies, President of IMU International Mathematical Union)

  Myung-Hwan Kim, President of KMS (Korean Mathematical Society)
- ICM Panel 3. **Mathematics is everywhere** by Christiane Rousseau http://www.youtube.com/v/r14n-p1uGM78

  Panelists

  Eduardo Colli, Universidade de São Paulo, Brazil

  Fidel Nemenzo, University of the Philippines, Philippines

  Konrad Polthier, Universität Freie Berlin, Germany
- IMAGINARY Panel

  Math communication for the future - a Vision Slam by Gert-Martin Greuel http://www.youtube.com/watch?v=ldzT7KlRuEg&feature=youtu.be

  Panelists

  Cedric Villani, Institut Henri Poincaré, France

  David Grünberg, International School of Lausanne, Switzerland

  Carla Cederbaum, MFO and University of Tübingen, Germany

  Hyungju Park, NIMS and POSTECH, South Korea
- IMU Panel 1

  Mathematical Massive Open Online Courses http://www.youtube.com/v/bRjkbmuCm20

  Moderator

  James Davenport, University of Bath, UK

  Panelists

  Bill Barton, The University of Auckland, New Zealand

  Robert Ghrist, University of Pennsylvania, USA

  Matti Pauna, University of Helsinki, Finland

  Angel Ruiz, Universidad de Costa Rica, Costa Rica
- ICM2014: Mathematics Education and Popularization of Mathematics

  Teaching and learning 'What is Mathematics' by Gunter M. Ziegler http://www.youtube.com/v/5lC7vQPTPdQ
- ICM2014: Mathematics Education and Popularization of Mathematics

  The internet and the popularization of mathematics by Etienne Ghys http://www.youtube.com/v/T3azqb7vynQ

Olivera Djokić, PhD

Teacher Education Faculty, Belgrade

Professional information

USEFUL WEB-LOCATIONS

Mathematics is as old as the humankind itself is. Mathematics has had essential role in improving science, engineering and philosophy since ancient times. It has evaluated from the simple counting, measuring and accounting, to systematic studying of objects and moving of the physical objects, through application of abstraction and imagination, logic, subsequently becoming wide, complex and very often abstract discipline.

Within the rubric useful web locations of the periodical *Teaching Innovations,* we are going to present the Internet resources from the field of Mathematics, that we consider to be useful for presenters, researchers, students, pupils and all the people interested in Mathematics.

Mathematical Institute of the SASA
http://www.sanu.ac.rs/sanunov/matematicki_institut.asp

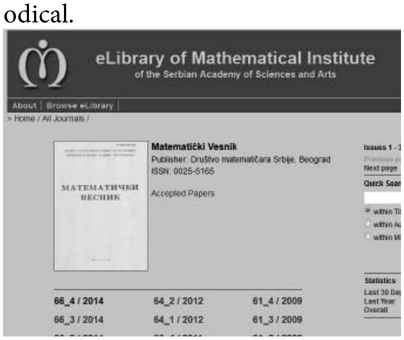
Mathematical Institute of the SASA, one of the institutes of the Serbian Academy of Sciences and Arts was founded in 1946. Mathematical Institute of the SASA does researches in the filed of Mathematics, Mechanics and Information, but actively participates in Mathematics promotion in wider public, among students, teachers and citizens.

Association of Mathematicians of Serbia
http://www.dms.rs/

Association of mathematicians of Serbia encourages and coordinates activities of its members towards realization mutual aims and tasks, in accordance with the Constitution, Law on social institutions and citizens' groups and other legislative regulations. Association contributes to improvement mathematical and computer sciences, their realization and popularization, and this is all in realization these aims and tasks. It also encourages scientific and professional work of its members, it helps scientific and professional research in the field of Mathematics, computer sciences and their application; it deals with the issues of taching mathematics and computer sciences in primary schools, higher schools and faculties and contributes to improvement of this kind of teaching; it deals with revealing, nourishing and developing gifted young mathematicians and computer programmers; it deals with the issues of status and protection of mathematics and mathematicians through certain organizational forms. The association reaches its aims through periodical meetings and gatherings at which they present scientific, professional and pedagogical works and papers about different issues of mathematical and compute sciences and their applications; issuing for their members periodicals and other publications in the field of Mathematics and other computer sciences; organizing different after school activities for young mathematicians and programmers (competitions, summer and winter schools, cycles of lectures, etc.); cooperation with scientific and educational institutions for Mathematics and Computer Science, suitable pedagogical institutions an other social organizations, cooperation with similar societies in the territory of Serbia and with similar societies in other countries; through active co-operation with syndicates, chambers and similar organizations of teachers, collecting reference books from Mathematics and computer science and their application. With the aim of realization common needs and interests, there are programmes and plans of work brought out in the Association, and creativity is encouraged.

Mathematical herald
http://elib.mi.sanu.ac.rs/pages/browse_publication.php?db=mv

Association of Mathematicians of Serbia has been issuing this periodical since 1949. Scientific papers containing original contribution to mathematical and computer sciences have been published in it. Papers are published in the following languages: English, Russian, French or German. Mathematical Herald has international editorial board and digital versions of the old issues of the periodical.



Apart from the stated periodical, the Association of Mathematicians of Serbia issues the following periodicals:

- *Matematicki list – periodical for Mathematics and Computer Science for high school students. Available at:* http://www.dms.org.rs/index.php?action=matematicki_list&change=true
- *Nastava matematike* – periodical for primary and secondary school teachers, as well as higher vocational schools and faculties. Available at: http://elib.mi.sanu.ac.rs/pages/browse_publication.php?db=nm
- *The Teaching of Mathematics* – research periodical in the field of Mathematics and computer science. Available at: elib.mi.sanu.ac.rs/journals/tm

Short history of Mathematics
http://elementarium.cpn.rs/elementi/kratka-istorija-matematike/

The site of the Centre for Science Promotion, which is in charge for science and promotion of science, offers brief history of Mathematics.

The following sites offer more information on history of Mathematics:

- The MacTutor History of Mathematics archive
  http://www-groups.dcs.st-and.ac.uk/~history/
- The story of mathematics
  http://www.storyofmathematics.com/
- History of Mathematics Web Sites
  http://homepages.bw.edu/~dcalvis/history.html
- Texts on the History of Mathematics
  http://aleph0.clarku.edu/~djoyce/mathhist/textbooks.html
- The British Society for the History of Mathematics
  http://www.dcs.warwick.ac.uk/bshm/
- The Canadian Society for the History and Philosophy of Mathematics
  http://www.cshpm.org/
- Material for the History of Statistics
  http://www.york.ac.uk/depts/maths/histstat/welcome.htm
- Teaching with Original Historical Sources in Mathematics
  http://math.nmsu.edu/~history/
- Images of Mathematicians on Postage Stamps
  http://members.tripod.com/jeff560/index.html
- Free Math eBooks Online
  http://www.techsupportalert.com/free-books-math

Archimedes
http://www.arhimedes.co.rs/

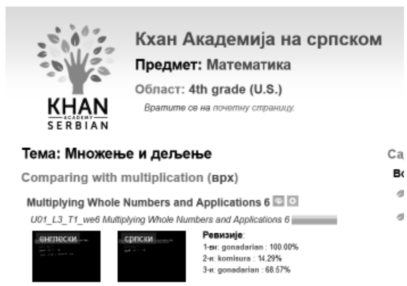
Mathematical association *Archimedes* (previously: Club of young

Mathematicians *Archimedes*) is a specialized professional society in the field of education and pedagogical work, which primarily gathers gifted young mathematicians and other mathematical and computer fans of different ages (primary and high school students, teachers and other adults who are dealing with Mathematics), and this is in the whole region of Serbia (mostly in Belgrade). It was founded on October 1st 1973 in Belgrade. There are two categories of members: a) adults, б) students and pupils (the young). There have been over 30 350 members so far (28 400 students and 1 950 teachers and other Mathematics, Computer science and Natural Sciences fans). The statute of the association proposes the aims, tasks, organization, management and the whole work. The Managing board consisting of eleven members, among which there are University professors, teachers from primary and high schools, associates from other institutions, manages it. Archimedes organises mathematical tournaments, seminars and courses. If you want to learn more about the club of young mathematicians, visit their web site.

Khan Academy in Serbian
http://khanacademy.rs/

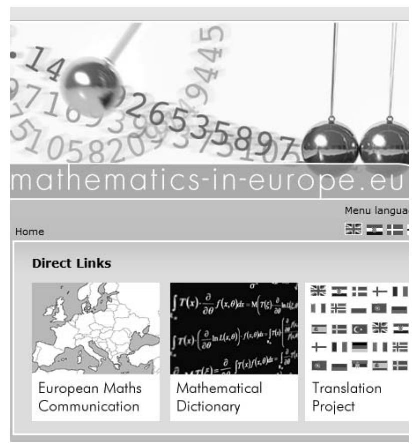
Han Academy is the Internet portal in which we can find educational contents in the form of video recordings, which localization in Serbian is in progress. Digital educational contents of Khan Academy can be used both in formal and informal education and this is represented in the motto of the academy: "Our mission is giving free education of the highest level to all the people, everywhere". In order to be acquainted with the contents of the portal, have a look at introductory video lessons about addition and subtraction, multiplication and division and geometry.

(Recommended location: https://www.khanacademy.org/math/)



Mathematics in Europe
http://www.mathematics-in-europe.eu/

Public comprehension of Mathematics is totally opposite to significance of that science in society. Many of our contemporaries consider Mathematics to be the field in which all significant results have been given long time ago, many centuries ago, and there is not much if any connections to real life. This is far from the truth. It is true that Mathematics is an important ingredient in our everyday life, that Mathematics is fascinating: mathematical problem can occupy your attention for days, months and even years; without Mathematics it could not be possible to comprehend how contemporary science describes the world: theory of relativity, quant mechanics etc.; they cannot be understood without the certain mathematical foundation. The aim of this web page, as the authors say, is to lessen the gap between the usual comprehension of Mathematics and truth. It addresses everyone who is interested in Mathematics: reporters, high school students, university students, teachers, professional Mathematicians, everyone who tries to find to way of raising public awareness of Mathematics. Nevertheless, we want to stress that our aim is not describing the "tough" scientific part of Mathematics. We want to present the contents, which can be understood, at least in most cases, without particular mathematical previous knowledge.



Inspirational mathematical lectures
http://www.ted.com/

TED (Technology, Education and Design) – non-profitable organisation devoted to spreading inspirational lectures, experience and ideas. At the site *TED.com,* we can access quality free lectures from different fields. We select four TED films, recommended by the Centre for Science Promotion, which were shown within manifestation "May – month of Mathematics":

- Jean-Baptiste Michel:
  The mathematics of history

https://www.youtube.com/watch?v=RkTE1LZ_tLk

- Conrad Wolfram: *Teaching kids real math with computers* ttps://www.youtube.com/watch?v=60OVlfAUPJg
- Marcus du Sautoy: *Symmetry, reality's riddle* https://www.youtube.com/watch?v=415VX3QX4cU
- Geoffrey West: *The surprising math of cities and corporations* https://www.youtube.com/watch?v=XyCY6mjWOPc

Math World
http://mathworld.wolfram.com

Math World is interactive mathematical encyclopaedia aimed at teachers, students, researchers and lovers of Mathematics. It covers all mathematical fields.

Famous mathematicians
http://famous-mathematicians.org/

Mathematics is the field in which many researchers are interested. They searched for the ways to understand the world referring to number and their contribution is immense. The given site offers a list of names of some of the researchers and their achievements.

History of Mathematics for Young Mathematicians
http://www.mathsisgoodforyou.com/

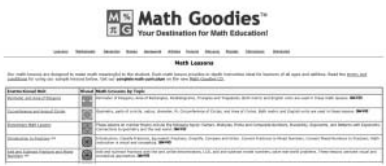
The site *Maths is good for you* was created in April 2005 with the aim of getting the young acquainted with history of Mathematics. It is aimed for students age 11 to 18 and teachers. The site is regularly updated and it is very organized.

Mathematics
http://www.math.com/

Site *Math.com* is devoted to Mathematics popularization and its understanding. It is for students, teachers, parents and all the people interested in Mathematics. *Math.com* offers unique experience through educational multimedia through play and research.

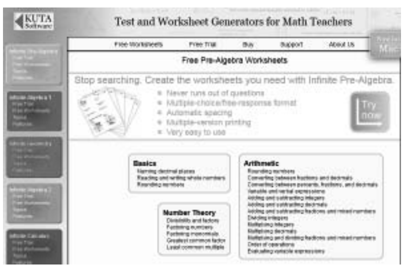
Mathematics
http://www.mathgoodies.com/

Math Goodies is one of the first free sites, which deal with teaching, and learning Mathematics and it was built in 1988. At this site, we can find over 500 pages with lessons and activities connected to teaching and learning Mathematics. The site is for students, teachers and parents.



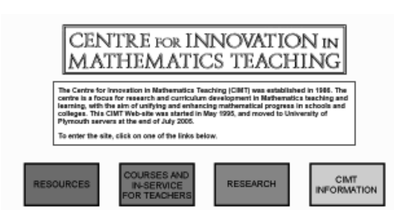
Site for teachers of Mathematics
https://www.kutasoftware.com/freeipa.html

Authors of the site, based on experience in teaching Mathematics, created the site with the aim of helping teachers to make classes of Mathematics efficient.



Centre for Innovation in Mathematics Teaching
http://www.cimt.plymouth.ac.uk/

The centre for innovation in mathematics teaching was founded in 1986 because of improving teaching Mathematics at schools and universities in Great Britain. One of the contributions of this centre is presenting and updating the location *Mathematics Enhancement Programme*.



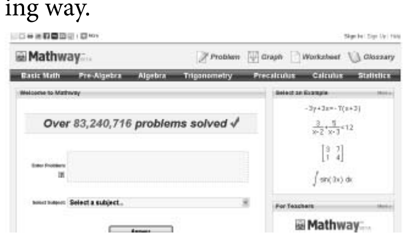
Considering the fact that the language of Mathematics is universal, the given materials can be useful to students and teachers all around the world. Most of the data are in PDF.

Mathematical games for children
http://www.kidsmathgamesonline.com/

Apart from the wide scope of free mathematical games, the site is full of different interesting digital educational resources, which motivate young mathematicians.

Mathematics
http://www.mathway.com/

Given location is for students, teachers and parents. We can find more than ten million mathematical problems from different fields, with the key. The site is simple to use, and contents are presented in an interesting way.



Mathematical museum in Gisen
http://www.mathematikum.de/

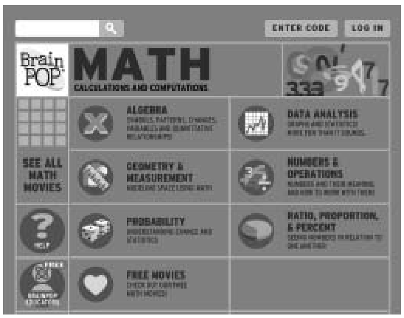
Mathematical museum in Gisen (Germany) was opened in 2002. It is proud of the fact that it is the first mathematical scientific centre in the world with more than one hundred fifty items for visitors of all ages and levels of education. Apart from fun exhibitions, M*athematikum* organises festival for children and lectures, which are being held every month.

Museum of Mathematics in New York
http://momath.org/

According to the words of the founder of the New York Museum of Mathematics, tendencies will be focused to improvement of understanding and comprehension of Mathematics. Dynamic exhibitions and programmes will provoke research and curiosity, subsequently revealing wonders of Mathematics. Museum activities will attract various visitors and get them acquainted with creative, humane and aesthetic nature of Mathematics that has been changing and improving all the time. As it is stated at the site, the idea for opening this museum appeared after closing a small museum of Mathematics in Long Island. After several meetings, it was concluded that in the whole USA, there was no mathematical museum, but that there is a great need for the programmes, which will enable Mathematics to be approached in an interesting way.

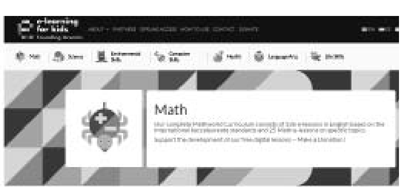
Science for children and adults
www.brainpop.com

The Internet location *Brain POP* is full of digital educational contents in the field of basic education. It is for children, parents and teachers. Data and explanations from the field of Mathematics are easily understandable owing to relevant and quality multimedia contents.



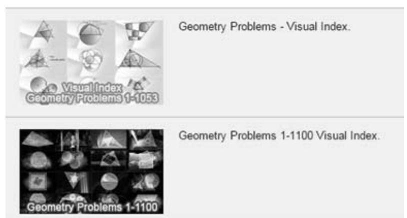
E-learning for kids
http://www.e-learningforkids.org/math/

E-learning for kids is a global, non-profitable organization devoted to open learning and fun on the Internet for children age 5 to 12. The site was founded in 2004 because of introduction innovation in teaching surrounding rich in ICT. There are digital materials at the site necessary for basic education and pedagogical work from natural and social sciences. It is for children, parents and teachers. From the field of Mathematics, there are over 336 analyzed teaching topics.



Geometry – step by step
http://agutie.homestead.com/files/index.html

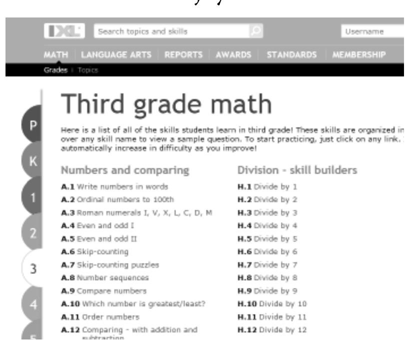
Geometry – step by step is a Canadian site, which has been awarded many times. It has been authentically created. There is a great number or animations, programmes and educational computer games devoted to Geometry. There is a chapter on geometrical problems on the site as well as their detailed solutions. Quizzes on the location offer practical techniques of solving geometry assignments, and connections to similar locations make contents of this location even richer.



Let's do mathematical exercises
http://www.ixl.com/

Doing mathematical exercises can be fun! The site *IXL.com* enables teachers and parents to follow their children's advancement and to motivate them by interactive games and quizzes, which are on the site.

The site needs dynamic surrounding which stimulates students to exercise and enjoy Mathematics.

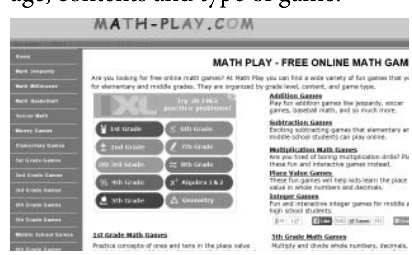


Educational computer games
http://www.abcya.com/

When we talk about educational computer games and activities for primary schools on the Internet, *ABC.com* is one of the leaders. All the educational games are free of charge. They include Mathematics, Language, Art and Basic Computer knowledge and skills.

Free mathematical games
http://www.math-play.com/

Digital games are well known to children and the young. Their engagement during the play, interactions often form new ways of learning and teaching. In recent years, digital games are more than before in the centre of the research of many fields and teachers, but more people deal with them in the industry of digital games. In the given site, we can find free mathematical games for primary school students, categorized by the age, contents and type of game.



Most visited web locations for educational mathematical games

http://www.mathchimp.com/

http://www.aaamath.com/

http://www.mathgametime.com/

http://www.coolmath4kids.com/

http://calculator.com/

http://www.numbernut.com/

http://www.toytheater.com/math.php

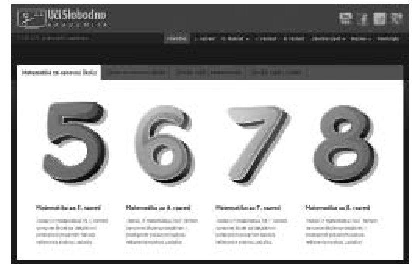
http://www.multiplication.com/games/all-games

http://www.mathsisfun.com/index.htm

http://www.arcademicskillbuilders.com/

Free Learning Academy
http://ucislobodno.com/

Learn free is the site with video instructions for solving mathematical tasks at the final exam of the middle school. The site attracted attention of students and parents and these points at the need for educational sites of this type.



Miroslava Ristic, PhD
Teacher Education Faculty, Belgrade

Стручне информације

КОРИСНЕ ВЕБ-ЛОКАЦИЈЕ

Математика је стара скоро колико и само човечанство. Од давнина, математика је од суштинског значаја за напредак у науци, инжењерству и филозофији. Она је еволуирала од простог бројања, мерења и обрачуна, преко систематичног проучавања облика и кретања физичких објеката, кроз примену апстракције, маште и логике, у данас широку, сложену и често апстрактну дисциплину.

У оквиру рубрике корисне веб-локације, часописа *Иновације у настави*, у овом тематском броју презентоваћемо интернет ресурсе из области математике за које сматрамо да могу бити од користи предавачима, истраживачима, студентима, ученицима, али и свима онима који су заинтересовани за математику.

Математички институт САНУ
http://www.sanu.ac.rs/sanunov/matematicki_institut.asp

Математички институт САНУ, један од института Српске академије наука и уметности, основан је 1946. године. Математички институт САНУ спроводи истраживања у области математике, механике и информатике, али такође активно учествује у промоцији математике у широј јавности, међу студентима, наставницима и грађанима.

Друштво математичара Србије
http://www.dms.rs/

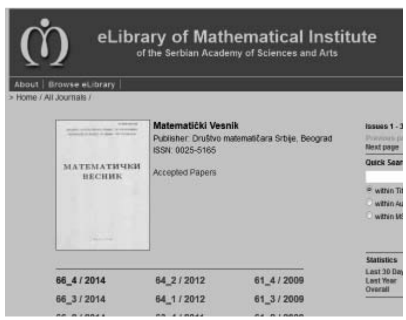
Друштво математичара Србије подстиче и координира активност својих чланова на остваривању заједничких циљева и задатака у складу са Уставом, Законом о друштвеним организацијама и удружењима грађана и другим законским прописима. У остваривању ових циљева и задатака Друштво: доприноси напретку математичких и рачунарских наука, њихових примена и популаризацији ових наука; подстиче на научни и стручни рад своје чланове, помаже научна и стручна истраживања у области математике, рачунарства и њихових примена; бави се питањима наставе математике и рачунарства у основним школама, у средњим школама, на вишим школама и на факултетима и доприноси унапређењу те наставе; бави се откривањем, неговањем и развијањем обдарених младих математичара и програмера; бави се питањима статуса и заштите математике и математичара кроз одговарајуће организационе форме. Своје циљеве Друштво постиже: периодичним скуповима и састанцима на којима се приказују научни, стручни и педагошки радови и реферати о разним проблемима математичких и рачунарских наука и њихових примена; издавањем за своје чланове часописа и других публикација у области математичких и рачунарских наука; организовањем разних видова ваннаставних активности за младе математичаре и програмере (такмичења, летњих и зимских школа, циклуса предавања и сл.); сарађивањем са научним и образовним институцијама за математику и рачунарство, одговарајућим просветно-педагошким институцијама и другим радним и друштвеним организацијама; сарађивањем са сродним друштвима на територији Србије и са сродним друштвима у другим земљама; кроз активну сарадњу са синдикатима, коморама и сличним организацијама просветних радника; прикупљањем литературе из математике и рачунарства и њихових примена. У циљу остваривања заједничких потреба и интереса у Друштву се доносе програми и планови рада, помаже и подстиче стваралаштво.

Математички весник
http://elib.mi.sanu.ac.rs/pages/browse_publication.php?db=mv

Овај научни часопис Друштво математичара Србије издаје од 1949. године. У њему се објављују научни радови који садрже оригиналне доприносе математичким и рачунарским наукама. Радови се објављују на једном од следећих језика: енглески, руски, француски или немачки. Математички весник има интернационалну редакцију и дигитализоване верзије старих годишта часописа.

Осим наведеног часописа, Друштво математичара Србије издаје и следеће часописе:

- *Математички лист – часопис за математику и рачунарство намењен ученицима средњих школа. Доступан на:* http://www.dms.org.rs/index.php?action=matematicki_list&change=true
- *Настава математике – часопис намењен наставницима и професорима основних и средњих школа, као и виших школа и универзитета. Доступан на:* http://elib.mi.sanu.ac.rs/pages/browse_publication.php?db=nm
- *The Teaching of Mathematics – истраживачки часопис у области наставе математике и рачунарства. Доступан на:* elib.mi.sanu.ac.rs/journals/tm



Кратка историја математике
http://elementarium.cpn.rs/elementi/kratka-istorija-matematike/

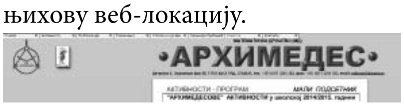
На сајту Центра за промоцију науке, који је надлежан за промоцију и популаризацију науке, можемо пронаћи кратку историју математике.

О историји математике можете више сазнати на следећим сајтовима:

- The MacTutor History of Mathematics archive http://www-groups.dcs.st-and.ac.uk/~history/
- The story of mathematics http://www.storyofmathematics.com/
- History of Mathematics Web Sites http://homepages.bw.edu/~dcalvis/history.html
- Texts on the History of Mathematics http://aleph0.clarku.edu/~djoyce/mathhist/textbooks.html
- The British Society for the History of Mathematics http://www.dcs.warwick.ac.uk/bshm/
- The Canadian Society for the History and Philosophy of Mathematics http://www.cshpm.org/
- Material for the History of Statistics http://www.york.ac.uk/depts/maths/histstat/welcome.htm
- Teaching with Original Historical Sources in Mathematics http://math.nmsu.edu/~history/
- Images of Mathematicians on Postage Stamps http://members.tripod.com/jeff560/index.html
- Free Math eBooks Online http://www.techsupportalert.com/free-books-math

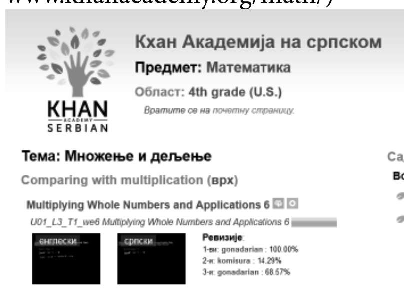
Архимедес
http://www.arhimedes.co.rs/

Математичко друштво *Архимедес* (раније: Клуб младих математичара *Архимедес*) јесте специјализовано стручно друштво у области образовања и васпитања, које окупља првенствено даровите младе математичаре и друге љубитеље математике и рачунарства разних узраста (ученици основних и средњих школа, студенти, наставници и други одрасли који се баве математиком), и то на целом подручју Републике Србије (највише у Београду). Основан је 1. октобра 1973. године у Београду. Има две категорије чланова: а) одрасли, б) ученици и студенти (подмладак). До сада је евидентирано преко 30 350 чланова (28 400 ученика и 1 950 наставника и других љубитеља математике, рачунарства и природних наука). Статутом Друштва утврђени су циљ и задаци, организација, управљање и целокупна његова делатност. Радом Друштва руководи Управни одбор од једанаест чланова, међу којима су универзитетски професори, наставници из основних и средњих школа, сарадници из других институција. *Архимедес* организује математичке турнире, семинаре и курсеве. Ако желите да сазнате нешто више о клубу младих математичара, посетите њихову веб-локацију.



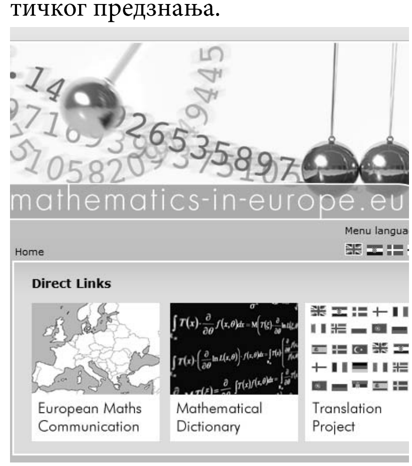
Кхан академија на српском
http://khanacademy.rs/

Кхан академија је интернет портал на којем се могу наћи образовни садржаји у облику видеоснимака, чија је локализација на српски језик управо у току. Дигитални образовни садржаји *Кхан академије* могу се користити и у формалном и у неформалном образовању, што одражава мото *Кхан академије*: „Наша мисија је пружање слободног образовања највишег нивоа свима, свугде." Како бисте се упознали са садржајем портала, погледајте уводне видео лекције из сабирања и одузимања, множења и дељења и геометрије. (Препорука за локацију: https://www.khanacademy.org/math/)



Математика у Европи
http://www.mathematics-in-europe.eu/

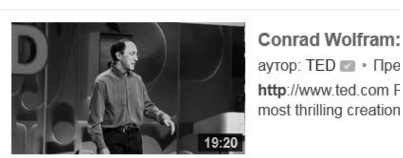
Јавно разумевање математике је у изразитој супротности са значајем те науке у друштву. Многи наши савременици сматрају да је математика поље у ком су сви битни резултати добијени одавно, пре више векова, и да има мало, ако уопште има, веза са стварним животом. То је далеко од истине. Истина је да је математика важан састојак у нашој свакодневици; да је математика фасцинантна: математички проблем може заокупити вашу пажњу данима, месецима, па чак и годинама; да без математике није могуће разумети како савремена наука описује свет: теорија релативности, квантна механика итд. не могу се разумети без одређене математичке подлоге. Циљ ове веб-странице, како кажу аутори, јесте да умањи расцеп између уобичајеног разумевања математике и стварне истине. Она се обраћа сваком кога занима математика: новинарима, средњошколцима, студентима, наставницима, професионалним математичарима, сваком ко тражи предлоге како да се повећа степен јавне свести о математици. Ипак, желимо да нагласимо да наш циљ није да опишемо „тврди" научни део математике. Желимо да представимо садржаје који се могу разумети, бар у већини случајева, без посебног математичког предзнања.



Инспиративна предавања из математике
http://www.ted.com/

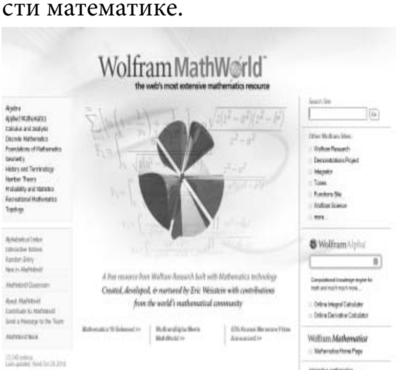
TED (Technology, Education and Design) – непрофитна је организација посвећена ширењу инспиративних предавања, искустава и идеја. На сајту *TED.com* можемо бесплатно приступити квалитетним предавањима из различитих области. Издвајамо четири TED филма, по препоруци Центра за промоцију науке, који су приказани у оквиру манифестације „Мај месец математике“:

- Jean-Baptiste Michel: *Историја математике* https://www.youtube.com/watch?v=RkTE1LZ_tLk
- Conrad Wolfram: *Деца уче стварну математику помоћу рачунара* https://www.youtube.com/watch?v=60OVlfAUPJg
- Marcus du Sautoy: *Симетрија, загонетка реалности* https://www.youtube.com/watch?v=415VX3QX4cU
- Geoffrey West: *Чудесна математика градова и корпорација* https://www.youtube.com/watch?v=XyCY6mjWOPc



Свет математике
http://mathworld.wolfram.com

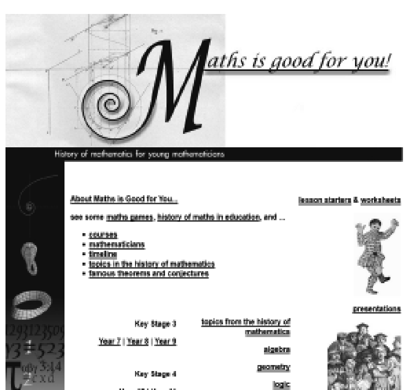
Свет математике је интерактивна математичка енциклопедија намењена наставницима, ученицима, истраживачима и љубитељима математике. Она покрива све области математике.



Познати математичари
http://famous-mathematicians.org/

Математика је област за коју су многи истраживачи показивали интересовање. Тражили су начине да схвате свет који се односи на број и њихови доприноси су драгоцени. На датом сајту можемо наћи списак имена неких истраживача и њихових достигнућа.

Историја математике за младе математичаре
http://www.mathsisgoodforyou.com/

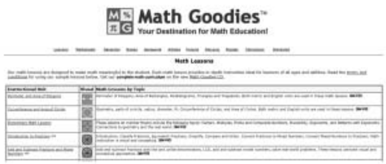


Сајт *Maths is good for you* креиран је у априлу 2005. године са циљем да се млади упознају са историјом математике. Намењен је ученицима узраста од једанаест до осамнаест година и наставницима. Сајт се редовно ажурира и прегледан је.

Математика
http://www.math.com/

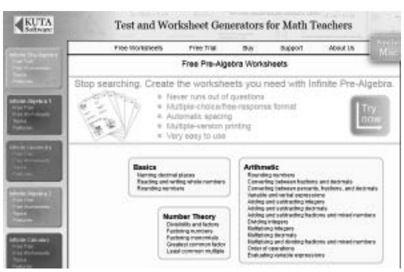
Сајт *Math.com* посвећен је популаризацији математике и њеном разумевању. Намењен је ученицима, родитељима, наставницима и свима заинтересованима за математику. *Math.com* нуди јединствено искуство путем образовне мултимедије која предшколце и ученике основних школа уводи у свет математике кроз игру и истраживање.

Математика
http://www.mathgoodies.com/



Math Goodies је један од првих бесплатних сајтова који се бави наставом и учењем математике, а постављен је 1988. године. На овом сајту данас можемо пронаћи преко петсто страна са лекцијама и активностима које су у вези са наставом и учењем математике, као и стручним чланцима из методике математике. Сајт је намењен ученицима, наставницима и родитељима.

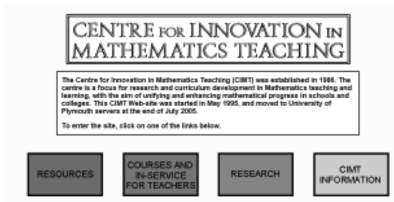
Сајт за наставнике математике
https://www.kutasoftware.com/freeipa.html



Аутори сајта, на основу година искуства у поучавању математике, креирали су сајт са намером да помогну наставницима да наставу математике учине ефективнијом и ефикаснијом.

Центар за унапређење наставе математике
http://www.cimt.plymouth.ac.uk/

Центар за унапређење наставе математике је основан 1986. године ради унапређења наставе математике у школама и на факултетима у Великој Британији. Један од доприноса овога центра је постављање и ажурирање локације *Mathematics Enhancement Programme.*



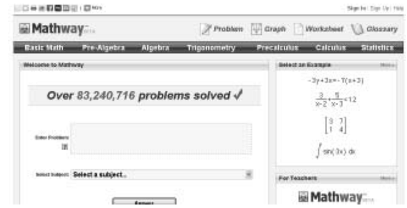
С обзиром на то да је језик математике универзалан, изложена грађа може бити од користи ученицима и наставницима широм света. Највећи део података је у PDF формату.

Математичке игре за децу
http://www.kidsmathgamesonline.com/

Осим широким спектром бесплатних математичких игара, сајт обилује и различитим занимљивим дигиталним образовним ресурсима који мотивишу младе математичаре.

Математика
http://www.mathway.com/

Дата локација намењена је ученицима, наставницима и родитељима. На њој можемо наћи преко десет милиона решених задатака из различитих математичких области. Сајт је једноставан за употребу, а садржаји су презентовани на занимљив начин.



Математички музеј у Гизену
http://www.mathematikum.de/

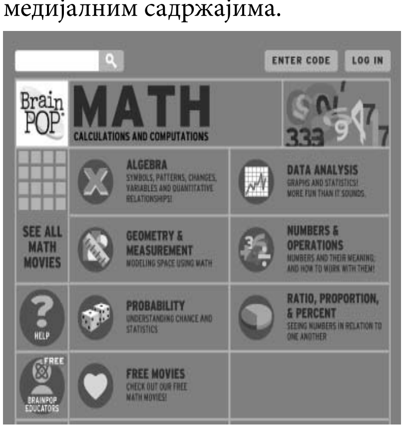
Математички музеј *Mathematikum* у Гизену (Немачка) отворен је 2002. године и поноси се чињеницом да је први математички научни центар у свету са преко сто педесет експоната за посетиоце свих узраста и нивоа образовања. Осим забавних сталних поставки, *Mathematikum* организује и фестивал за децу, као и предавања која се одржавају сваког месеца.

Музеј математике у Њујорку
http://momath.org/

По речима оснивача њујоршког музеја математике, тежња ће бити усмерена на унапређивање разумевања и доживљавања математике. Динамичне поставке и програми ће подстицати истраживање и радозналост и откривати чуда математике. Музејске активности ће широку и разноврсну публику упознавати са креативном, хуманом, естетском природом математике, која се стално мења и унапређује. Како наводе на сајту, идеју за отварање музеја добили су по затварању малог музеја математике на Лонг Ајленду. После неколико састанака, установљено је да у читавим Сједињеним Америчким Државама не постоји ниједан математички музеј, али да за програмима који би омогућавали да се математици приступи на занимљив начин постоји велика потражња.

Наука за децу и одрасле
www.brainpop.com

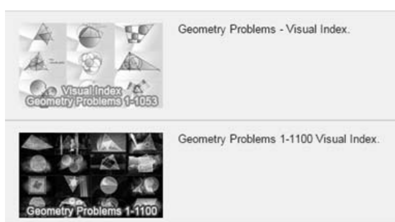
Интернет локација *Brain POP* обилује дигиталним образовним садржајима из области базичног образовања. Намењена је пре свега деци, затим родитељима и учитељима. Подаци и објашњења из области математике су лако разумљиви захваљујући релевантним и квалитетним образовним мултимедијалним садржајима.



Е-учење за децу
http://www.e-learningforkids.org/math/

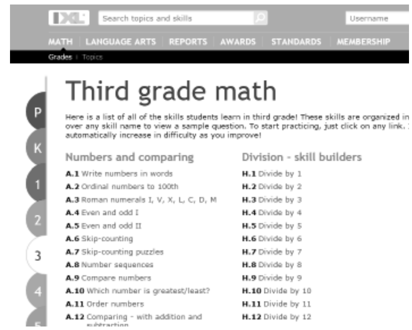
Е-учење за децу је глобална, непрофитна фондација посвећена отвореном учењу и забави на интернету за децу узраста од пет до дванаест година. Сајт је основан 2004. године ради увођења иновација у наставно окружење богато ИКТ-ом. На сајту се налазе дигитални материјали неопходни за базично образовање и васпитање из природних и друштвених наука. Намењен је деци, родитељима и наставницима. Из области математике постоји преко триста тридесет шест обрађених наставних тема.

Геометрија – корак по корак
http://agutie.homestead.com/files/index.html



Геометрија – корак по корак је канадски сајт који је више пута награђиван. Сајт је аутентично уређен. Има велики број анимација, програма и образовних рачунарских игара посвећених геометрији. На сајту постоји одељак са геометријским проблемима, као и њиховим детаљним решењима. Квизови на локацији нуде практичне технике решавања геометријских задатака, а везе ка сличним локацијама употпуњују богат садржај ове локације.

Вежбајмо математику
http://www.ixl.com/



Вежбање математике може бити забавно! Сајт *IXL.com* омогућава наставницима и родитељима да прате напредак своје деце и ученика и да их мотивишу путем интерактивних игара и квизова који се налазе на сајту.

Сајт нуди динамичко окружење које стимулише ученике да вежбају и уживају у математици.

Образовне рачунарске игре
http://www.abcya.com/

Када су у питању образовне рачунарске игре и активности за основце на интернету, *ABCya.com* је један од лидера. Све образовне игре су бесплатне. Обухватају области као што су: математика, језик, уметност и основе рачунарских знања и вештина.

Бесплатне математичке игре
http://www.math-play.com/

Дигиталне игре су деци и младима добро познате. Њихов ангажман при игрању, као и интеракције, често стварају нове начине учења али и поучавања. Последњих година дигиталне игре су више него раније у центру пажње истраживача различитих профила и наставника, али још увек се њима много више баве запослени у индустрији дигиталних игара. На датом сајту можемо наћи бесплатне математичке игре за основце које су разврстане према узрасту, садржају и врсти игре.

Најпосећеније веб-локације са образовним математичким играма

http://www.mathchimp.com/

http://www.aaamath.com/

http://www.mathgametime.com/

http://www.coolmath4kids.com/

http://calculator.com/

http://www.numbernut.com/

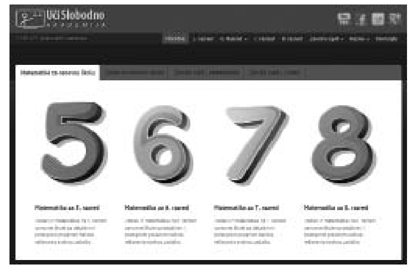
http://www.toytheater.com/math.php

http://www.multiplication.com/games/all-games

http://www.mathsisfun.com/index.htm

http://www.arcademicskillbuilders.com/

Учи слободно Академија
http://ucislobodno.com/



Учи слободно је сајт са видео-упутствима за решавање задатака из математике на завршном испиту, тј. малој матури. Сајт је привукао пажњу ученика и родитеља, што указује на потребу за образовним сајтовима овог типа.

др Мирослава Ристић
Учитељски факултет, Београд

Teaching Innovations

GENERAL INFORMATION

Teaching Innovations is a scientific periodical issued by the Teacher Education Faculty, University of Belgrade. It includes theoretical and systematic review papers and original research papers related to sciences and scientific disciplines dealing with the teaching process at all levels of pedagogical and educational work with the aim of its improvement and modernisation. *Teaching Innovations* is intended to provide support to researchers, and inspiration to practitioners to find optimal solutions and efficient strategies for introducing innovations in pre-school, primary, secondary and tertiary education, including life-long learning.

The periodical is issued quarterly.

PAPER SUBMISSION GUIDELINES

The following categories of scientific papers are published in the *Teaching Innovations* periodical:

1. Original scientific paper (reporting previously unpublished results of the author's original research based on the IMRAD (Introduction, Methods, Results and Discussion) scientific method scheme);

2. Systematic review (presenting original, detailed and critical review of the issue under study including the author's personal contribution, proved by self-citation);

3. Short scientific paper (original scientific paper which summarises the results of one's original research work or work that is still in progress);

4. Review paper (the known findings and results of original research are presented with the aim of spreading information and knowledge as well as their application in praxis).

Apart from scientific and review papers, the *Teaching Innovations* periodical publishes translations of papers, informative reviews and general reviews (of books, computer programmes, educational software, scientific events, etc.), as well as profession-related information.

Manuscripts should be sent by e-mail and are not returned. The electronic address of the editorial board is: inovacije@uf.bg.ac.rs. Papers can be submitted in Serbian, English, Russian or French. Papers positively assessed by the reviewers will be published in the Periodical in the language in which they were written. The authors who want their paper to be published in a foreign language (English, Russian or French), must have it translated into the language of their choice.

All papers are anonymously reviewed by two component reviewers. **The author is obliged to inform the editorial board in writing about any changes made in the text (number of the page which includes the changes with all the changes highlighted) according to the reviewers' comments and recommendations.** Upon that, the decision regarding publication is made, which the author is informed of within three months.

The paper submitted for publication should conform with the *Teaching Innovations* style sheet in order to be taken into consideration for reviewing. **Papers which do not comply with the outlined style sheet will be returned to the author (authors) for revision.**

STYLE SHEET

1. Font. The paper should be written in Microsoft Word, font Times New Roman size 12. Paragraphs: font – Normal, spacing – 1.5, the first line automatically indented. (Col 1)

2. Volume. The full volume of systematic reviews and original research articles is up to 16 pages (36 000 characters); short scientific papers, critiques, polemics and discussions, as well as review papers or translated papers up to 8 pages (about 15 000 characters); and event reports and short reviews up to 2 - 3 pages (about 3800-5600 characters). The editor has the right to accept longer papers if the research requirements are such.

3. General information about the authors. Name, middle name (initial only) and surname are given in the heading, affiliation in the line below. The third line should include home address or Institution address and the birth year (the birth year is not published, but it is used for paper classification at the National library of Serbia). The author's name should be accompanied with a footnote stating the author's e-mail address. If there are several authors, only one (preferably the first author's) address should be provided. If the paper is based on a doctoral thesis, the footnote should include the title of the thesis, place and faculty where the viva took place. Papers resulting from research projects should include the project title and registry number, the funding organisation and institution of its application. Position: left.

4. Title of the paper. Three lines below the name. Font: Times New Roman, 12, bold; position: centre.

5. Summary. It can be 150–300 words long, and should be given at the beginning of the paper, one line below the title. It should state the aim of the paper, applied research methods, the most significant results and conclusions. **The editorial board provides translation of the summaries into English or translation of extended summaries from other languages into Serbian.** The editorial board does not provide translations of full papers into foreign languages.

6. Key words. They are stated below the summary. There should be up to five words in *italics*, in standard letters, separated by a comma (with a full stop behind the last one).

7. The text body. Papers should be written concisely, in a comprehensible style and in a logical order. As a rule, it includes the introductory part with a clearly stated aim or the main problem of the paper, description of methodology, presentation and discussion of the results, and a conclusion with suggestions for further research or praxis.

8. References in the text. Literature used is referred to in brackets and included in the body of the text, not in a footnote. Surnames of foreign authors used in the text body are quoted in the original form or are phonetically transcribed in Serbian, accompanied by the original in brackets with the year of publishing included. For example: Mejer (Meyer, 1987). If the paper was written by two authors, surnames of both are stated; in the case of more than two authors, the surname of the first author is stated, followed by "et al."

9. Citations. No matter how long, the citation should be followed by a reference to the page number.

10.Tables, graphs, schemas, pictures. Tables and graphs should be in Word or a similar compatible programme. Each table, graph or schema must be comprehensible without reading the text, i.e. it must be marked with an ordinal number, title and caption (not longer than one line) and the legend (explanation of marks, codes and abbreviations). Pictures should be prepared in the electronic form in the 300dpi resolution and jpg format. Tables, graphs, schemas and pictures should be inserted in proper places in the text. Showing the same data in table and graph formats is unacceptable. Illustrations taken from other sources (books, journals) must be quoted with the source. Apart from that, a written consent from the copyright owner should be obtained and submitted to the editorial office.

11. Statistical analysis results. Results of statistical interpretations should be presented in the following way: F=25.35, df=1,9, p< . 001 or F(1,9)=25,35, p< .001 (as common in the statistics of pedagogical and psychological research).

12. Footnotes and abbreviations. Not allowed, except in special cases.

13. List of references. The end of the text should be followed by a list of references quoted in the text, in alphabetical order and in the following way:

BOOK

Sahlberg, P. (2011). *Finnish Lessons. What can the world learn from educational change in Finland?* New York: Teacher College Press, Teachers College, Columbia University.

PAPER IN A PERIODICAL

Haslam, A. A., Jetten, J., Postmes, T. and Haslam, C. (2009). Social Identity, Health and Well-Being: An Emerging Agenda for Applied Psychology. *Applied Psychology*, 58 (1), 1-23.

CHAPTER IN A BOOK or REVIEW IN A BOOK OF PROCEEDINGS

Zgaga, P., Devjak, T., Vogrinc, J. and Repac I. (2001). National report - Slovenia. In: Zgaga, P. (ed.). *The Prospect of Teacher Education in South-east Europe* (527–570). Ljubljana: University of Ljubljana, Faculty of Education.

WEB DOCUMENTS

Kallestad, J. and Olweus, D. (2003). Predicting Teachers and Schools Implementation of the Olweus Bullying Prevention Program: A Multilevel Study. *Prevention and Treatment*, Vol. 6, No. 2. Retrieved May 18, 2000. from www: http://www.vanguard.edu/psychology/apa.pdf.

The reference list should only include references cited in the text body or those analysed in a review paper.

When the same author is cited several times, this should be done following the sequence of years in which the papers were published. If several cited papers were written by the same author and published in the same year, references should be marked by letters next to the year of issuance, for example 1999a, 1999b... Citing unpublished works should be avoided.

Иновације у настави

Часопис *Иновације у настави* научни је часопис који издаје Учитељски факултет Универзитета у Београду. У њему објављујемо теоријске, прегледне и оригиналне истраживачке радове из наука и научних дисциплина које третирају наставни процес на свим нивоима васпитања и образовања у циљу његовог унапређења и модернизације. Циљ је да *Иновације* буду подршка истраживачима, а инспирација практичарима у проналажењу оптималних решења и ефикасних стратегија за увођење иновација у настави од предшколског васпитања преко основношколске, средњошколске и универзитетске наставе до целоживотног образовања.

Часопис излази четири пута годишње.

УПУТСТВО АУТОРИМА

У часопису *Иновације у настави* објављујемо научне чланке који припадају следећим категоријама:

1. изворни научни чланак (у коме се износе претходно необјављени резултати сопствених истраживања научним методом према шеми IMRAD (Introduction, Methods, Results and Discussion));

2. прегледни научни чланак (рад који садржи оригиналан, детаљан и критички приказ истраживачког проблема у коме је аутор остварио одређен допринос, видљив на основу аутоцитата);

3. кратки научни чланак (изворни научни чланак који сажима резултате изворног истраживачког дела или дела које је још у току);

4. стручни чланак (у коме се саопштавају позната сазнања и резултати изворних истраживања, са намером ширења информација и сазнања, као и њихове примене у пракси).

Осим научних и стручних радова, у часопису *Иновације у настави* објављујемо преведене радове, информативне прилоге и приказе (књига, рачунарских програма, образовних софтвера, научних догађаја и др.), као и стручне информације.

Рукописи се шаљу електронском поштом и не враћају се. Електронска адреса редакције је: inovacije@uf.bg.ac.rs. Аутори могу послати радове на српском, енглеском, руском или француском језику. Сви радови који добију позитивне рецензије биће објављени у часопису на језику на ком су написани. Уколико аутори желе да рад буде објављен у часопису на страном језику (енглеском, руском или француском), неопходно је да га преведу на језик који су одабрали.

Сви радови се анонимно рецензирају од стране два компетентна рецензента. **Аутор је дужан да у писменој форми редакцију упозна са свим изменама које је начинио у тексту (број странице на којој се налази измена и означавање места на коме је промена извршена), у складу са примедбама и пре-**

порукама рецензената. Након тога, уређивачки одбор доноси одлуку о објављивању. О томе обавештава аутора у року од три месеца.

Рад приложен за објављивање треба да буде припремљен према стандардима часописа *Иновације у настави* како би био укључен у процедуру рецензирања. **Неодговарајуће припремљени рукописи биће враћени аутору (одн. ауторима) на дораду.**

СТАНДАРДИ ЗА ПРИПРЕМУ РАДА

Фонт. Рад треба да буде написан у текст процесору Microsoft Word, фонтом Times New Roman, величине 12 тачака. Параграфи: фонт – Normal, проред – 1.5, први ред – увучен аутоматски (Col 1).

Обим. Прегледни и истраживачки радови могу бити дужине до једног ауторског табака (16 страна, око 36.000 знакова), кратки научни чланци, критике, полемике и осврти, као и стручни и преведени радови до 8 страна (око 15.000 знакова); извештаји и прикази до 2–3 стране (приближно 3800–5600 знакова). Уредник задржава право да објави обимније радове када изражавање научног садржаја захтева већи простор.

Општи подаци о ауторима. Име, средње слово и презиме аутора наводи се у првом реду, а у следећем се даје институција у којој ради. Испод тога треба навести адресу становања или институције у којој је аутор запослен и годину рођења (година рођења се не објављује, али се користи приликом класификације радова у Народној библиотеци Србије). Позиција: left. Поред свог имена аутор додаје фусноту, у чијем садржају на дну странице наводи своју електронску адресу. Ако је аутора више, треба дати само адресу једног, обично првог. Уколико рад потиче из докторске дисертације, у фусноти уз наслов треба да стоји и назив тезе, место и факултет на којем је одбрањена. За радове који потичу из истраживачких пројеката треба навести назив и број пројекта, финансијера и институцију у којој се реализује.

Наслов рада. Три реда испод имена. Фонт: Times New Roman, 12, bold; позиција: center.

Резиме. Може бити дужине 150–300 речи, налази се на почетку рада, један ред испод наслова. Садржи циљ рада, примењене методе истраживања, најзначајније резултате и закључке. **Редакција обезбеђује превођење резимеа на енглески језик или превођење проширених резимеа са других језика на српски језик.** Редакција не обезбеђује превод радова у целини на стране језике.

Кључне речи. Наводе се иза резимеа. Треба да их буде до пет, пишу се италик стандардним словима и одвојене су зарезом (иза последње стоји тачка).

Основни текст. Радове треба писати језгровито, разумљивим стилом и логичким редом. Он, по правилу, укључује уводни део, који се завршава одређењем циља или проблема рада, опис методологије, приказ добијених резултата, дискусију резултата и закључак са препорукама за даља истраживања или за праксу.

Референце у тексту. На литературу се упућује у загради у самом тексту, а не у фусноти. Имена страних аутора у тексту се наводе у српској транскрипцији (према одредбама у важећем Правопису), а затим се у загради наводи изворно, уз годину публиковања рада. Пример: Мејер (Meyer, 1987). Када постоје два аутора рада, наводе се презимена оба, док се у случају већег броја аутора наводи презиме првог и скраћеница „и сар.“ уколико је реч о раду на српском, или „et al.“ уколико је реч о раду на страном језику.

Цитати. Сваки цитат, без обзира на дужину, треба да прати референца са бројем стране. Пример: (Meyer, 1987: 38).

Табеле, графикони, схеме, слике. Треба да буду сачињени у Word-у или неком њему компатибилном програму. Табеле из статистичких пакета треба „пребацити" у Word. Свака табела, схема, слика и сваки графикон морају бити разумљиви и без читања текста, односно, морају имати редни број, наслов (прецизан, не дужи од једног реда) и легенду (објашњења ознака, шифара и скраћеница). Слике треба припремити у електронској форми са резолуцијом од 300dpi и у формату jpg. Табеле, схеме, слике и графикони треба да буду распоређени на одговарајућа места у тексту. Приказивање истих података табеларно и графички није прихватљиво. За илустрације преузете из других извора (књига, часописа) аутор је дужан да упути на извор. Осим тога, потребно је да прибави и достави редакцији писмено одобрење власника ауторских права.

Резултати статистичке обраде. Треба да буду дати на следећи начин: F=25.35, df=1,9, p<.001 или F(1,9)=25,35, p<.001 (како је уобичајено у статистици педагошких и психолошких истраживања).

Фусноте и скраћенице. Нису дозвољене, осим у изузетним случајевима.

Списак литературе. На крају текста треба приложити списак литературе (по азбучном реду ако је рад писан ћирилицом, односно по абецедном реду ако је рад писан латиницом) на следећи начин:

КЊИГА

Sahlberg, P. (2011). *Finnish Lessons. What can the world learn from educational change in Finland?* New York: Teacher College Press, Teachers College, Columbia University.

Радовановић, И., Радовић, В. Ж. и Тадић, А. (2009). *Иновације у настави – библиографија радова (1983–2008)*. Београд: Учитељски факултет.

ПОГЛАВЉЕ У КЊИЗИ

Хавелка, Н. (2001). Уџбеник и различите концепције образовања и наставе. У: Требјешанин, Б. и Лазаревић. Д. (ур.). *Савремени основношколски уџбеник* (31–58). Београд: Завод за уџбенике и наставна средства.

ЧЛАНАК У ЧАСОПИСУ

Гашић Павишић, С. (2009). Знања и уверења будућих учитеља о вршњачком насиљу међу ученицима. *Иновације у настави, 22* (4), 71–84.

ПРИЛОГ У ЗБОРНИКУ

Ристић, М. (2009). Вредновање знања ученика у систему е-учења. *Иновације у основношколском образовању – вредновање* (522–530). Београд: Учитељски факултет.

ВЕБ-ДОКУМЕНТИ

Kallestad, J. and Olweus, D. (2003). Predicting Teachers and Schools Implementation of the Olweus Bullying Prevention Program: A Multilevel Study. *Prevention and Treatment*, Vol. 6, No. 2. Retrieved May 18, 2000. from www: http://www.vanguard.edu/psychology/apa.pdf.

У списку литературе наводе се само референце на које се аутор позива или које је анализирао у прегледном чланку.

Када се исти аутор наводи више пута, поштује се редослед година у којима су радови публиковани. Уколико се наводи већи број радова истог аутора публикованих у истој години, радови треба да буду означени словима уз годину издања нпр. 1999а, 1999б... Навођење необјављених радова није пожељно.